Probability measures on a space, “two-valued” topologies and localic toposes of probability measures

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Probability measures on a space :

Definition. – *Let X be a set.*

Let $\mathcal{U}$ be a family of subsets of X which is stable by $\begin{cases} \text{finite intersections,} \\ \text{countable unions.} \end{cases}$

A probability measure on $(X, \mathcal{U})$ is an application

$$\mu : \mathcal{U} \longrightarrow [0,1]$$

such that

$$\begin{cases} \bullet \quad \mu(\emptyset) = 0 \text{ and } \mu(X) = 1, \\ \bullet \quad \mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V) \text{ for any } U, V \in \mathcal{U}, \\ \bullet \quad \mu(U) = \sup_{n \in \mathbb{N}} \mu(U_n) \\ \qquad \text{for any increasing sequence of } U_n \in \mathcal{U},\ n \in \mathbb{N}, \text{ whose union is } U = \bigcup_{n \in \mathbb{N}} U_n. \end{cases}$$

Remark. – The family $\overline{\mathcal{U}}$ of subsets of X which are arbitrary unions of elements of $\mathcal{U}$ is stable by $\begin{cases} \text{arbitrary unions,} \\ \text{finite intersections.} \end{cases}$

It is a topology on X.

The concept of μ-negligible difference :

Definition. – *Let μ be a probability measure on a space $(X,\mathcal{U})$.*
The difference between two ordered elements of $\mathcal{U}$

$$U' \subseteq U$$

is said to be μ-negligible if, for any $\varepsilon > 0$, there is an element U'' of $\mathcal{U}$ such that

$$U \subseteq U' \cup U'' \text{ and } \mu(U'') < \varepsilon.$$

Lemma. –

(i) *If the difference between two elements $U' \subseteq U$ of $\mathcal{U}$ is μ-negligible, the same is true of the difference $U' \cap V \subseteq U \cap V$ for any $V \in \mathcal{U}$.*

(ii) *If differences $U'' \subseteq U'$ and $U' \subseteq U$ are μ-negligible, the same applies to the difference $U'' \subseteq U$.*

(iii) *For any sequence of ordered pairs of elements of $\mathcal{U}$*

$$U'_n \subseteq U_n\,, \quad n \in \mathbb{N}\,,$$

whose differences are μ-negligible,
the same applies to the difference

$$\bigcup_{n\in\mathbb{N}} U'_n \subseteq \bigcup_{n\in\mathbb{N}} U_n\,.$$

Grothendieck topology associated with a notion of negligible :

Definition. – *Let $\mathcal{U}$ be an ordered set equipped with*

- *a sup $\bigvee$ of countable families,*
- *an inf $\wedge$ of finite families, distributive with respect to $\bigvee$.*

Let $\mathcal{N}$ be a family of ordered pairs $U' \leq U$ of elements of $\mathcal{U}$, such that :

(1) *Whenever $U' \leq U$ is in $\mathcal{N}$, then for any $V \in \mathcal{U}$, $U' \wedge V \leq U \wedge V$ is still in $\mathcal{N}$.*

(2) *If $U'' \leq U'$ and $U' \leq U$ are in $\mathcal{N}$, then $U'' \leq U$ is still in $\mathcal{N}$.*

(3) *If $(U'_n \leq U_n)_{n \in \mathbb{N}}$ are in $\mathcal{N}$, then $\bigvee_{n \in \mathbb{N}} U'_n \leq \bigvee_{n \in \mathbb{N}} U_n$ is still in $\mathcal{N}$.*

Then we define a Grothendieck topology $J_{\mathcal{N}}$ on $\mathcal{U}$, seen as a cartesian category, by deciding that a family of morphisms

$$U_i \leq U\,, \quad i \in I\,,$$

is covering if it contains a countable subfamily

$$U_{i_n} \leq U\,, \quad n \in \mathbb{N}\,,$$

such that the ordered pair

$$\bigvee_{n \in \mathbb{N}} U_{i_n} \leq U$$

is an element of $\mathcal{N}$.

Grothendieck topologies associated with notions of negligible :

Lemma. – *Let $\mathcal{U}$ be an ordered set equipped with*

- *a sup $\bigvee$ of countable families,*
- *an inf $\wedge$ of finite families, distributive with respect to $\bigvee$.*

Then a Grothendieck topology J on $\mathcal{U}$ is the topology $J_{\mathcal{N}}$ associated with a notion of "negligible difference" $\mathcal{N}$ on ordered pairs

$$U' \leq U \text{ of elements of } \mathcal{U},$$

if and only if it satisfies the following conditions :

(1') *A family of morphisms of $\mathcal{U}$ seen as a category*

$$U_i \leq U\,, \quad i \in I\,,$$

is J-covering if and only if it contains a countable J-covering subfamily.

(2') *For any countable family of elements of U*

$$(U_n)_{n\in\mathbb{N}} \quad \text{with} \quad \bigvee_{n\in\mathbb{N}} U_n = U\,,$$

the countable family of morphisms

$$U_n \longrightarrow U\,, \quad n \in \mathbb{N}\,,$$

is J-covering.

The topos associated with a notion of negligible difference :

Corollary. –
An ordered set $(\mathcal{U}, \leq)$
which admits ($\bigvee$ *countable,* $\wedge$ *finite distributive)*
and which is equipped with a notion $\mathcal{N}$ *of negligible difference*

of ordered pairs $U' \leq U$

defines

a site $(\mathcal{U}, J_{\mathcal{N}})$

and so

a topos $\widehat{\mathcal{U}}_{\mathcal{N}}$

endowed with a Cartesian canonical functor

$$\ell : \mathcal{U} \longrightarrow \widehat{\mathcal{U}}_{\mathcal{N}} \, .$$

Corollary. –
In particular, a probability measure μ *on some* $(X, \mathcal{U})$
defines a notion $\mathcal{N}_{\mu}$ *of negligible difference of ordered pairs*

$$U' \subseteq U$$

and therefore a topos

$$\widehat{\mathcal{U}}_{\mathcal{N}_{\mu}}$$

endowed with a Cartesian canonical functor

$$\ell : \mathcal{U} \longrightarrow \widehat{\mathcal{U}}_{\mathcal{N}_{\mu}} \, .$$

Points of toposes and flat functors :

For a notion $\mathcal{N}$ of negligible difference on
($\mathcal{U}$, $\leq$, $\bigvee$ countable, $\wedge$ finite distributive),
the category of points of the associated topos $\mathrm{pt}\,(\widehat{\mathcal{U}_{\mathcal{N}}})$
identifies with the category of functors $x^* : \mathcal{U} \longrightarrow \mathrm{Set}$
which are

- flat, i.e. Cartesian (since $\mathcal{U}$ is Cartesian),
- $J_{\mathcal{N}}$-continuous.

Lemma. – *Let ($\mathcal{U}$, $\leq$, $\wedge$ finite) be a Cartesian ordered set, which in particular admits a greater element X.*

(I) The flat functors (i.e. Cartesian functors)

$$x^* : \mathcal{U} \longrightarrow \mathrm{Set}$$

are indexed by subfamilies $\mathcal{P}$ of $\mathcal{U}$ such that

- *$\mathcal{P}$ contains X and is stable by $\wedge$,*
- *for every $U \leq V$, we have $V \in \mathcal{P}$ if $U \in \mathcal{P}$.*

(ii) The functor $x^*_{\mathcal{P}}$ associated with such a subfamily $\mathcal{P}$ is

$$\begin{cases} U \longmapsto \{\bullet\} & \text{if} \quad U \in \mathcal{P}\,, \\ U \longmapsto \emptyset & \text{if} \quad U \notin \mathcal{P}\,. \end{cases}$$

The points of the topos $\widehat{\mathcal{U}}_{\mathcal{N}}$:

Proposition. –

Let $\mathcal{N}$ be a notion of negligible difference on

$$(\mathcal{U}, \leq, \bigvee \text{ countable}, \wedge \text{ finite distributive}).$$

Let x^* be a cartesian functor

$$x^* = x^*_{\mathcal{P}} : \mathcal{U} \longrightarrow \mathrm{Set}$$

defined by a subfamily $\mathcal{P} \subseteq \mathcal{U}$ which is stable by

$$\begin{cases} \bullet & \text{finite inf } \wedge \text{ ,} \\ \bullet & \text{switch to larger elements.} \end{cases}$$

Then x^* is $J_{\mathcal{N}}$-continuous if and only if, for every $U \in \mathcal{U}$ and every countable family

$$U_n \leq U\,, \quad n \in \mathbb{N},$$

such that the difference

$$\bigcup_{n \in \mathbb{N}} U_n \leq U$$

is in $\mathcal{N}$, we have

$$U \in \mathcal{P}$$

if and only if there exists $n \in \mathbb{N}$ such that

$$U_n \in \mathcal{P}\,.$$

The special case of families of subsets of a space :

Lemma. – *Let us assume that* $(\mathcal{U}, \leq, \bigvee$ *countable,* $\wedge$ *finite)*
is a family of subsets of a space X,
which we suppose stable by countable unions $\bigvee$ *and finite intersections* $\wedge$.
Then :

(I) Any element $x \in X$ *defines a cartesian functor*

$$x^* = x^*_{\mathcal{P}} : \mathcal{U} \longrightarrow \mathrm{Set}$$

by $\mathcal{P} = \mathcal{P}_x = \{U \in \mathcal{U} \mid x \in U\}$.

(ii) This functor is $J_{\mathcal{N}}$*-continuous if and only if,*
for any ordered pair of elements of $\mathcal{U}$

$$U' \subseteq U,$$

such that $x \in U$ *and* $x \notin U'$*, the difference*

$$U' \subseteq U$$

cannot be in $\mathcal{N}$.

Remark. – If $\mathcal{N}$ is defined by a probability measure μ on $\mathcal{U}$,
the condition of (ii) is verified if, for any pair

$U' \subseteq U$ such that $x \in U$ and $x \notin U'$, one has $\mu(U') < \mu(U)$.

Spaces of sequences and incidence frequencies :

Let X be a set,
and $X^{\mathbb{N}}$ be the space of sequences of elements of X

$$x_\bullet = (x_n)_{n\in\mathbb{N}}\,.$$

Definition. –
For any sequence $x_\bullet = (x_n)_{n\in\mathbb{N}}$,
the sequence of the incidence frequencies of a subset $U \subseteq X$ *in* $x_\bullet$ *is*

$$p_n^U(x_\bullet) = \frac{\#\{0 \le k \le n \mid x_k \in U\}}{n+1} \in [0,1]\,, \quad n \in \mathbb{N}\,.$$

Definition. –
For any sequence $x_\bullet = (x_n)_{n\in\mathbb{N}}$,
the lower and upper limit frequencies of a subset $U \subseteq X$ *in* $x_\bullet$ *are*

$$p_-^U(x_\bullet) = \liminf_{n\mapsto+\infty} p_n^U(x_\bullet) = \lim_{n\mapsto+\infty} \inf_{k\ge n} p_k^U(x_\bullet)$$

and

$$p_+^U(x_\bullet) = \limsup_{n\mapsto+\infty} p_n^U(x_\bullet) = \lim_{n\mapsto+\infty} \sup_{k\ge n} p_k^U(x_\bullet)\,.$$

Subspaces of sequences defined by limit incidence frequencies :

Definition. –
For any subset $U \subseteq X$
and any element $q \in [0,1]$,
we have two associated subspaces of $X^{\mathbb{N}}$

$$P^U_{\geq q}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_-(x_\bullet) \geq q\}$$

and

$$P^U_{\leq q}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_+(x_\bullet) \leq q\}.$$

Remarks. –

(i) We have $x_\bullet \in P^U_{\geq q}(X^{\mathbb{N}})$
if and only if, for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} \mid p^U_n(x_\bullet) < q - \varepsilon\} \text{ is finite.}$$

(ii) Likewise, we have $x_\bullet \in P^U_{\leq q}(X^{\mathbb{N}})$
if and only if, for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} \mid p^U_n(x_\bullet) > q + \varepsilon\} \text{ is finite.}$$

Lattice of subspaces defined by limit incidence frequencies :

Definition. – *Let X be a set.*
Let $\mathcal{U}$ be a family of subsets of X
stable by finite intersections and countable unions.
Let Q be a dense subset of $[0,1]$.
We will then denote

$$\mathcal{U}_{\mathbb{N}}$$

the family of subsets of $X^{\mathbb{N}}$
which can be written as countable unions
of finite intersections of subsets of the form

$$P^U_{\geq q}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_-(x_\bullet) \geq q\}$$

or

$$P^U_{\leq q}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_+(x_\bullet) \leq q\}$$

with $U \in \mathcal{U}$ and $q \in Q$.

Remark. – Therefore $\mathcal{U}_{\mathbb{N}}$ is the smallest family of subsets of $X^{\mathbb{N}}$ which contains the

$$P^U_{\geq q}(X^{\mathbb{N}}) \text{ and } P^U_{\leq q}(X^{\mathbb{N}}), \quad U \in \mathcal{U}, q \in Q,$$

and which is stable by finite intersections and countable unions.

Inclusion relations between subspaces defined by limit frequencies :

We consider as previously a family $\mathcal{U}$
of subsets of a set X.

Lemma. –

(i) *For any subset $U \in \mathcal{U}$ of X and any elements $q_1 \leq q_2$ of $Q \subseteq [0,1]$, we have the inclusion relation*

$$P^U_{\geq q_1}(X^{\mathbb{N}}) \supseteq P^U_{\geq q_2}(X^{\mathbb{N}})$$

and

$$P^U_{\leq q_1}(X^{\mathbb{N}}) \subseteq P^U_{\leq q_2}(X^{\mathbb{N}}) .$$

(ii) *For any subsets $U \subseteq V$ of X belonging to $\mathcal{U}$ and any element q of $Q \subseteq [0,1]$, we have the inclusion relation*

$$P^U_{\geq q}(X^{\mathbb{N}}) \subseteq P^V_{\geq q}(X^{\mathbb{N}})$$

and

$$P^U_{\leq q}(X^{\mathbb{N}}) \supseteq P^V_{\leq q}(X^{\mathbb{N}}) .$$

Exclusion relations between subspaces defined by limit frequencies :

We still consider a family $\mathcal{U}$
of subsets of a set X.

Lemma. –
*For any subset $U \in \mathcal{U}$ of X
and any elements $q < q'$ of $Q \subseteq [0,1]$,
we have the exclusion relation*

$$P^U_{\leq q}(X^{\mathbb{N}}) \cap P^U_{\geq q'}(X^{\mathbb{N}}) = \emptyset\,.$$

Proof. –
This follows from the definitions

$$P^U_{\leq q}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_+(x_\bullet) \leq q\}\,,$$
$$P^U_{\geq q'}(X^{\mathbb{N}}) = \{x_\bullet \in X^{\mathbb{N}} \mid p^U_-(x_\bullet) \geq q'\}$$

since $p^U_+(x_\bullet)$ and $p^U_-(x_\bullet)$
are the upper and lower limits of the same sequence

$$P^U_n(x_\bullet)\,, \quad n \in \mathbb{N}\,.$$

The property of additivity of incidence frequencies :

Lemma. –
For any sequence $x_\bullet \in X^{\mathbb{N}}$ of elements of X and for any subsets U, V of X, we have for any $n \in \mathbb{N}$ the formula

$$p_n^U(x_\bullet) + p_n^V(x_\bullet) = p_n^{U\cup V}(x_\bullet) + p_n^{U\cap V}(x_\bullet) \, .$$

Proof. –
Indeed, we have for any subset U of X

$$p_n^U(x_\bullet) = \frac{1}{n+1} \cdot \sum_{0\leq k\leq n} \mathbb{I}_U(x_k) \, ,$$

denoting

$$\begin{array}{cccl} \mathbb{I}_U & : & X & \longrightarrow & \{0,1\} \\ & & x & \longmapsto & \begin{cases} 1 & \text{if} \quad x \in U \, , \\ 0 & \text{if} \quad x \notin U \, , \end{cases} \end{array}$$

and we observe that the functions

$$\mathbb{I}_U \, , \; \mathbb{I}_V \, , \; \mathbb{I}_{U\cup V} \, , \; \mathbb{I}_{U\cap V}$$

are linked by the formula

$$\mathbb{I}_U + \mathbb{I}_V = \mathbb{I}_{U\cup V} + \mathbb{I}_{U\cap V} \, .$$

Translation of additivity for subspaces defined by incidence frequencies :

Corollary. – *For any subsets* $U, V \in \mathcal{U}$ *of* X *and any elements* $q_1, q_2, q_3, q_4 \in Q \subseteq [0,1]$ *linked by the formula* $q_1 + q_2 = q_3 + q_4$, *we have the inclusion relations*

$$\begin{cases} P^U_{\geq q_1}(X^{\mathbb{N}}) \cap P^V_{\geq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\leq q_3}(X^{\mathbb{N}}) \subseteq P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}}), \\ P^U_{\leq q_1}(X^{\mathbb{N}}) \cap P^V_{\leq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\geq q_3}(X^{\mathbb{N}}) \subseteq P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}}), \end{cases}$$

$$\begin{cases} P^U_{\geq q_1}(X^{\mathbb{N}}) \cap P^V_{\geq q_2}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}}) \subseteq P^{U\cap V}_{\geq q_3}(X^{\mathbb{N}}), \\ P^U_{\leq q_1}(X^{\mathbb{N}}) \cap P^V_{\leq q_2}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}}) \subseteq P^{U\cap V}_{\leq q_3}(X^{\mathbb{N}}), \end{cases}$$

$$\begin{cases} P^U_{\geq q_1}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\leq q_3}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}}) \subseteq P^V_{\leq q_2}(X^{\mathbb{N}}), \\ P^U_{\leq q_1}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\geq q_3}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}}) \subseteq P^V_{\geq q_2}(X^{\mathbb{N}}), \end{cases}$$

$$\begin{cases} P^V_{\geq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\leq q_3}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}}) \subseteq P^U_{\leq q_1}(X^{\mathbb{N}}), \\ P^V_{\leq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\geq q_3}(X^{\mathbb{N}}) \cap P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}}) \subseteq P^U_{\geq q_1}(X^{\mathbb{N}}). \end{cases}$$

Expressing the "law of large numbers" :

Theorem. –
*Let $\mathcal{U}$ be a family of subsets of a set X which is stable by finite intersections and countable unions.
Let $\mu : \mathcal{U} \to [0,1]$ be a probability measure .
Let $\mathcal{U}_{\mathbb{N}}$ be the family of subspaces of $X^{\mathbb{N}}$ which are countable unions of finite intersections of subspaces of the form*

$$P^U_{\geq q}(X^{\mathbb{N}}) \quad \text{or} \quad P^U_{\leq q}(X^{\mathbb{N}}) \quad \text{with } U \in \mathcal{U} \text{ and } q \in Q \subseteq [0,1] \,.$$

Then :

(i) *The measure μ on $\mathcal{U}$ induces a product measure $\mu_{\mathbb{N}}$ on $\mathcal{U}_{\mathbb{N}}$.*

(ii) *The measure $\mu_{\mathbb{N}}$ induces a notion of "negligible difference" $\mathcal{N}_{\mathbb{N}}$ such that, for any subset $U \in \mathcal{U}$ and any $q \in Q \subseteq [0,1]$, we have*
- *$P^U_{\leq q}(X^{\mathbb{N}})$ is negligible if $q < \mu(U)$,*
- *$P^U_{\geq q}(X^{\mathbb{N}})$ is negligible if $q > \mu(U)$,*
- *the difference $P^U_{\leq q}(X^{\mathbb{N}}) \subseteq X^{\mathbb{N}}$ is negligible if $q \geq \mu(U)$,*
- *the difference $P^U_{\geq q}(X^{\mathbb{N}}) \subseteq X^{\mathbb{N}}$ is negligible if $q \leq \mu(U)$.*

Consequence for the relationship between probability measures and Grothendieck topologies :

We consider as before a family $\mathcal{U}$ of subsets of X which is stable by finite intersections and countable unions.
We still denote $\mathcal{U}_{\mathbb{N}}$ the family of subspaces of $X^{\mathbb{N}}$ which are countable unions of finite intersections of subspaces of the form

$$P^U_{\geq q}(X^{\mathbb{N}}) \quad \text{or} \quad P^U_{\leq q}(X^{\mathbb{N}}) \quad \text{with } U \in \mathcal{U} \text{ and } q \in Q \subseteq [0,1]\,.$$

Then :

Corollary. –

(i) *A measure μ on $\mathcal{U}$ induces a product measure $\mu_{\mathbb{N}}$ on $\mathcal{U}_{\mathbb{N}}$.*

(ii) *The measure $\mu_{\mathbb{N}}$ induces a notion of “negligible difference” $\mathcal{N}_\mu$ on the ordered pairs of elements of $\mathcal{U}_{\mathbb{N}}$.*

(iii) *The knowledge of this notion $\mathcal{N}_\mu$ of “negligible difference” is equivalent to that of the Grothendieck topology J_μ on $\mathcal{U}_{\mathbb{N}}$ it defines.*

(iv) *It is also equivalent to knowing the subtopos $(\widehat{\mathcal{U}_{\mathbb{N}}})_{J_\mu}$ of $\widehat{\mathcal{U}_{\mathbb{N}}}$.*

(v) *The knowledge of $\mathcal{N}_\mu$ or of the topology J_μ is enough to reconstruct the measure μ on $\mathcal{U}$.*

Consequences independent of the choice of measure :

We still consider the family $\mathcal{U}$ of subsets of X
and the family $\mathcal{U}_{\mathbb{N}}$ of subspaces of $X^{\mathbb{N}}$
which is associated with it by the consideration
of limit incidence frequencies.

Corollary. –
For any measure μ of $\mathcal{U}$,
the notion of negligible difference $\mathcal{N}_\mu$ on $\mathcal{U}_{\mathbb{N}}$
which is induced by the product measure $\mu_{\mathbb{N}}$
satisfies the following property :

$$\begin{cases} \text{For any subset } U \in \mathcal{U} \\ \text{and any elements } q \geq q' \text{ of } Q \subseteq [0,1], \\ \text{the difference} \\ P^U_{\leq q}(X^{\mathbb{N}}) \cup P^U_{\geq q'}(X^{\mathbb{N}}) \subseteq X^{\mathbb{N}} \\ \text{is negligible.} \end{cases}$$

An expression of the compatibility of measures with countable unions :

We still consider the family $\mathcal{U}$ of subsets of X
and the family $\mathcal{U}_{\mathbb{N}}$ of subspaces of $X^{\mathbb{N}}$ which is associated with it.

Corollary. –
Suppose the dense subset $Q \subseteq [0,1]$ is countable.
For any measure μ on $\mathcal{U}$,
the induced notion $\mathcal{N}_\mu$ of negligible difference on $\mathcal{U}_{\mathbb{N}}$
satisfies the following property :

For any increasing sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$, with

$$U = \bigcup_{n\in\mathbb{N}} U_n\,,$$

and for any element $p \in [0,1]$, the difference between elements of $\mathcal{U}_{\mathbb{N}}$

$$\bigcup_{\substack{n\in\mathbb{N},\, q\in Q \\ q>p}} P^{U_n}_{\geq q}(X^{\mathbb{N}}) \subseteq \bigcup_{\substack{q\in Q \\ q>p}} P^{U}_{\geq q}(X^{\mathbb{N}})$$

is negligible.

Proof. –

- If $p \geq \mu(U)$, then all parts involved are negligible.
- If $p < \mu(U)$,then there exists $n \in \mathbb{N}$ and $q \in Q$ such that $p < q < \mu(U_n) \leq \mu(U)$.
  It follows that the difference $P^{U_n}_{\geq q}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$ is negligible.

The question of characterizing Grothendieck topologies associated with measures :

We recall that $\mathcal{U}$ is a family of subsets of a set X,
which is stable by finite intersections and by countable unions.
We denoted

$$\mathcal{U}_{\mathbb{N}} \quad \text{the family of subspaces of } X^{\mathbb{N}}$$

which are countable unions of finite intersections
of subspaces of the form

$$P^{U}_{\geq q}(X^{\mathbb{N}}) \qquad \text{or} \qquad P^{U}_{\leq q}(X^{\mathbb{N}})$$

with $U \in \mathcal{U}$ and $q \in Q$.
Here, Q is a subset of $[0,1]$ such that

- Q is countable,
- Q is dense In $[0,1]$,
- for any elements $q_1, q_2, q_3 \in Q$ and $q \in [0,1]$, we have

$$q \in Q \quad \text{if} \quad q_1 + q_2 = q_3 + q \, .$$

Question.– *How to characterize the Grothendieck topologies J on $\mathcal{U}_{\mathbb{N}}$, corresponding to a notion of negligible difference $\mathcal{N}$, which are associated with probability measures μ on $\mathcal{U}$?*

Statement of the characterization of topologies associated with measures :

Proposition. – *A Grothendieck topology J on $\mathcal{U}_{\mathbb{N}}$ is associated with a probability measure μ on $\mathcal{U}$ if and only if it corresponds to a notion of "negligible difference" $\mathcal{N}$ such that :*

(1) *$X^{\mathbb{N}}$ is not negligible.*

(2) *For any elements $q \geq q'$ of $Q \subseteq [0,1]$ and any $U \in \mathcal{U}$, the difference*

$$P^U_{\leq q}(X^{\mathbb{N}}) \cup P^U_{\geq q'}(X^{\mathbb{N}}) \subseteq X^{\mathbb{N}}$$

is negligible.

(3) *For any increasing sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$, with $U = \bigcup_{n\in\mathbb{N}} U_n$, and for any element $q \in Q$, the difference*

$$\bigcup_{n\in\mathbb{N}, q'\in Q, q'>q} P^{U_n}_{\geq q'}(X^{\mathbb{N}}) \subseteq \bigcup_{q'\in Q, q'>q} P^U_{\geq q'}(X^{\mathbb{N}})$$

is negligible.

(4) *For any $q \in Q \subseteq [0,1]$ and any $U \in \mathcal{U}$, we have*

$$\begin{cases} \bullet & \text{either } P^U_{\geq q}(X^{\mathbb{N}}) \text{ is negligible,} \\ \bullet & \text{or the difference } P^U_{\geq q}(X^{\mathbb{N}}) \subset X^{\mathbb{N}} \text{ is negligible.} \end{cases}$$

Identification of the measure :

We want to construct a measure μ on $\mathcal{U}$
from the topology J associated with a notion of negligible $\mathcal{N}$
which satisfies properties (1), (2), (3), (4) of the proposition.
It is naturally defined as follows :

Definition. –
For any subset $U \in \mathcal{U}$, we define

$$\mu(U) = \inf\{q \in Q \mid P^U_{\geq q}(X^{\mathbb{N}}) \text{ is negligible}\}.$$

Remarks. –
It follows from this definition and from property (4) :

(i) $P^U_{\geq q}(X^{\mathbb{N}})$ is negligible for any $q > \mu(U)$.

(ii) The difference

$$P^U_{\geq q}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

is negligible for any $q < \mu(U)$,
and therefore also for $q = \mu(U)$ is $\mu(U) \in Q$.

Statement and proof of the symmetric property :

Lemma. – *For any subset $U \in \mathcal{U}$, we have :*

(i) $P^U_{\leq q}(X^{\mathbb{N}})$ *is negligible for any* $q < \mu(U)$.

(ii) *The difference*

$$P^U_{\leq q}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

is negligible for any $q > \mu(U)$ *and also for* $q = \mu(U)$ *if* $\mu(U) \in Q$.

Proof. – We define $\mu_U = \sup\{q \in Q \mid P^U_{\leq q}(X^{\mathbb{N}}) \text{ is negligible}\}$.

- The intersections

  $$P^U_{\leq q}(X^{\mathbb{N}}) \cap P^U_{\geq q'}(X^{\mathbb{N}})$$

  are empty if $q < q'$.
  This implies that $P^U_{\leq q}(X^{\mathbb{N}})$ is negligible if $q < \mu(U)$ and so $\mu_U \geq \mu(U)$.
- The differences

  $$P^U_{\leq q}(X^{\mathbb{N}}) \cup P^U_{\geq q'}(X^{\mathbb{N}}) \subseteq X^{\mathbb{N}}$$

  are negligible if $q \geq q'$. This implies that the differences

  $$P^U_{\leq q}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

  are negligible if $q > \mu(U)$, and so $\mu_U \leq \mu(U)$.

The growth property of the measure :

Lemma. – *For any ordered pair $U_1 \subseteq U_2$ of $\mathcal{U}$, we have*

$$\mu(U_1) \leq \mu(U_2)\,.$$

Proof. – We have by definition

$$\mu(U_1) = \inf\{q \in Q \mid P^{U_1}_{\geq q}(X^{\mathbb{N}}) \text{ is negligible}\},$$

$$\mu(U_2) = \inf\{q \in Q \mid P^{U_2}_{\geq q}(X^{\mathbb{N}}) \text{ is negligible}\}.$$

The conclusion follows from the fact that the inclusion relation

$$U_1 \subseteq U_2$$

implies the inclusion relation

$$P^{U_1}_{\geq q}(X^{\mathbb{N}}) \subseteq P^{U_2}_{\geq q}(X^{\mathbb{N}})$$

for any $q \in Q$. It follows indeed that

$$P^{U_1}_{\geq q}(X^{\mathbb{N}}) \text{ is negligible}$$

$$\text{if} \qquad P^{U_2}_{\geq q}(X^{\mathbb{N}}) \text{ is negligible}.$$

The property of additivity of the measure :

Lemma. – *For all elements $U, V \in \mathcal{U}$, we have*

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)\,.$$

Proof. – For all elements $q_1, q_2, q_3, q_4 \in Q$ such that

$$q_1 + q_2 = q_3 + q_4\,,$$

we have the inclusions

$$P^U_{\geq q_1}(X^{\mathbb{N}}) \cap P^V_{\geq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\leq q_3}(X^{\mathbb{N}}) \subseteq P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}})\,,$$

$$P^U_{\leq q_1}(X^{\mathbb{N}}) \cap P^V_{\leq q_2}(X^{\mathbb{N}}) \cap P^{U\cap V}_{\geq q_3}(X^{\mathbb{N}}) \subseteq P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}})\,.$$

This implies :

- The difference

  $$P^{U\cup V}_{\geq q_4}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

  is negligible if $q_1 < \mu(U)$, $q_2 < \mu(V)$, $q_3 > \mu(U \cap V)$, and so $\mu(U \cup V) \geq \mu(U) + \mu(V) - \mu(U \cap V)$.
- The difference

  $$P^{U\cup V}_{\leq q_4}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

  is negligible if $q_1 > \mu(U)$, $q_2 > \mu(V)$, $q_3 < \mu(U \cap V)$, and so $\mu(U \cup V) \leq \mu(U) + \mu(V) - \mu(U \cap V)$.

Compatibility of the measure with countable increasing unions :

Lemma. – *For any increasing sequence $(U_n)_{n\in\mathbb{N}}$ of subsets $U_n \in \mathcal{U}$, with $U = \bigcup_{n\in\mathbb{N}} U_n$, we have*

$$\mu(U) = \sup_{n\in\mathbb{N}} \mu(U_n)\,.$$

Proof. – We already know that $\mu(U) \geq \mu(U_n)$ for any $n \in \mathbb{N}$.
We know on the other hand that for any $q \in Q$, the difference

$$\bigcup_{n\in\mathbb{N},q'\in Q,q'>q} P^{U_n}_{\geq q'}(X^{\mathbb{N}}) \subseteq \bigcup_{q'\in Q,q'>q} P^{U}_{\geq q'}(X^{\mathbb{N}})$$

is negligible. Moreover, if $q < \mu(U)$, the difference

$$\bigcup_{q'\in Q,q'>q} P^{U}_{\geq q'}(X^{\mathbb{N}}) \subset X^{\mathbb{N}}$$

is also negligible.
Thus, there exists $n \in \mathbb{N}$ and $q' > q$ such that

$$P^{U_n}_{\geq q'}(X^{\mathbb{N}})$$

is not negligible. This implies

$$\mu(U_n) \geq q' > q\,.$$

The conclusion follows as $q < \mu(U)$ can be chosen arbitrarily close.

The concept of "two-valued" topos :

Definition. –
A topos $\mathcal{E}$ is called "two-valued"
if the only two subobjects of its terminal object 1 *are*

$$\begin{cases} 1 \text{ itself,} \\ \text{the initial object } \emptyset. \end{cases}$$

Remark. –
If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ is the classifying topos
of some first-order geometric theory $\mathbb{T}$,
it is "two-valued" if and only if
the theory $\mathbb{T}$ is "complete" in the sense that,
for any geometric formula φ without free variable
written in the signature Σ of φ,
we have

- either φ is "provably true", i.e.
  $\top \vdash \varphi$ is $\mathbb{T}$-provable,
- or φ is "provably false", i.e.
  $\varphi \vdash \bot$ is $\mathbb{T}$-provable.

A reformulation of the notion of measure in terms of "two-valued" toposes :

We still consider a family $\mathcal{U}$ of subsets of a set X,
which is stable by finite intersections and countable unions.
We still note $\mathcal{U}_{\mathbb{N}}$ the ordered family of subspaces of $X^{\mathbb{N}}$
which are countable unions of finite intersections
of subspaces of $X^{\mathbb{N}}$ of the form

$$P^U_{\geq q}(X^{\mathbb{N}}) \quad \text{or} \quad P^U_{\leq q}(X^{\mathbb{N}}) \quad \text{with} \quad U \in \mathcal{U} \;\text{ and }\; q \in Q\,.$$

Here, Q is a countable and dense subset of $[0,1]$,
stable under the relation $q_1 + q_2 = q_3 + q_4$ of $[0,1]^4$.

Definition. – *Let $J_{\mathbb{N}}$ be the smallest Grothendieck topology of $\mathcal{U}_{\mathbb{N}}$ for which :*

(1) *Any countable union $P = \bigcup\limits_{n \in \mathbb{N}} P_n$ of subspaces $P_n \in \mathcal{U}_{\mathbb{N}}$ is covered by the family of the P_n's.*

(2) *For any elements $q \geq q'$ of Q and any $U \in \mathcal{U}$, $X^{\mathbb{N}}$ is covered by $P^U_{\leq q}(X^{\mathbb{N}})$ and $P^U_{\geq q'}(X^{\mathbb{N}})$.*

(3) *For any union $U = \bigcup\limits_{n \in \mathbb{N}} U_n$ of an increasing sequence of subsets $U_n \in \mathcal{U}$, and any $q \in Q$, the countable family of the $P^{U_n}_{\geq q'}(X^{\mathbb{N}})$, $n \in \mathbb{N}$, $q' > q$, covers $\bigcup\limits_{q' \in Q, q' > q} P^U_{\geq q'}(X^{\mathbb{N}})$.*

Reformulation of the equivalence between measures and topologies :

Proposition. –
Let $\mathcal{E}_{\mathbb{N}}$ be the topos of sheaves on the site

$$(\mathcal{U}_{\mathbb{N}}, J_{\mathbb{N}})$$

consisting in the ordered family $\mathcal{U}_{\mathbb{N}}$,
seen as a category,
and endowed with the Grothendieck topology $J_{\mathbb{N}}$.
Then the equivalence between probability measures μ on $\mathcal{U}$
and Grothendieck topologies J_μ on $\mathcal{U}_{\mathbb{N}}$

$$\mu \longleftrightarrow J_\mu$$

induces a one-to-one correspondence between

- *probability measures μ on $\mathcal{U}$,*
- *subtoposes $\mathcal{E}_\mu$ of $\mathcal{E}_{\mathbb{N}}$ which are "two-valued".*

Verification of this reformulation of the equivalence :

Considering a subtopos of $\mathcal{E}_{\mathbb{N}} = \widehat{(\mathcal{U}_{\mathbb{N}})}_{J_{\mathbb{N}}}$
is equivalent to considering a Grothendieck topology

$$J \supseteq J_{\mathbb{N}} \quad \text{on} \quad \mathcal{U}_{\mathbb{N}} \, .$$

According to the previous proposition,
it suffices to prove that if a topology $J \supseteq J_{\mathbb{N}}$
defines a two-valued topos,
then any covering family of morphisms of $\mathcal{U}_{\mathbb{N}}$

$$P_i \subseteq P \, , \quad i \in I \, ,$$

contains a countable covering subfamily.
In fact, any P or P_i, $i \in I$, covers the whole of $X^{\mathbb{N}}$
or is covered by the empty family.
If some P_i covers $X^{\mathbb{N}}$, it a fortiori covers P.
If, on the contrary, all P_i are covered by the empty family,
it is the same with P.
So, P admits in both cases a subcovering consisting in
at most one element of the family $(P_i)_{i \in I}$.

Points of a topos and "two-valued" subtoposes :

We recall :

Lemma. – *Consider a site* $(\mathcal{C}, J)$.

(i) *Any topos morphism* $\mathcal{E} \xrightarrow{f=(f^*,f_*)} \widehat{\mathcal{C}}_J$
canonically factors as the composite $\mathcal{E} \twoheadrightarrow \widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J$
of a surjective morphism $\mathcal{E} \twoheadrightarrow \widehat{\mathcal{C}}_{J'}$ *and an embedding* $\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J$.
This embedding part is defined by the topology $J' \supseteq J$ *on* $\mathcal{C}$ *for which a family of morphisms of* $\mathcal{C}$

$$(X_i \longrightarrow X)_{i \in I}$$

is covering if its transform by the functor

$$\rho = f^* \circ \ell : \mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{f^*} \mathcal{E}$$

is globally epimorphic.

(ii) *If* $\mathcal{E} = \mathrm{Set}$*, the topology* J' *of* $\mathcal{C}$ *defined by a point*

$$\mathrm{Set} \xrightarrow{p} \widehat{\mathcal{C}}_J$$

is necessarily "two-valued".

Proof of (ii). – Any subobject of the terminal object 1 of $\widehat{\mathcal{C}}_{J'}$ is transformed by p^* in a subobject of $\{\bullet\}$, which is $\{\bullet\}$ or $\emptyset$.

“Two-valued” subtoposes and points of localic toposes :

To any topos $\mathcal{E}$, we can associate the distributive lattice O of the subobjects of the terminal object 1 of $\mathcal{E}$:
Indeed, finite intersections $\wedge$ and arbitrary unions $\bigvee$ of subobjects of 1 are always defined in $\mathcal{E}$, and $\wedge$ is distributive relatively to $\bigvee$.
The ordered set O seen as a category, and endowed with the topology defined by $\bigvee$, defines a topos $\widehat{O}_{\bigvee}$ endowed with a morphism $\mathcal{E} \longrightarrow \widehat{O}_{\bigvee}$.
The topos $\mathcal{E}$ is said “localic” if this is an isomorphism.

Lemma. – *If $\mathcal{E}$ is a localic topos, any “two-valued” subtopos of $\mathcal{E}$ corresponds to a point of $\mathcal{E}$.*

Proof. – Let J be a topology on O which defines a “two-valued” subtopos of $\mathcal{E}$. Associate with any object $(X \hookrightarrow 1)$ of O

$$X \longmapsto \begin{cases} \emptyset & \text{if } X \hookrightarrow 1 \text{ is not } J\text{-covering,} \\ \{\bullet\} & \text{otherwise.} \end{cases}$$

This defines a point of the topos $\mathcal{E} \xrightarrow{\sim} \widehat{O}_{\bigvee}$.

The topos of probability measures :

We still consider a family $\mathcal{U}$ of subsets of a set X,
which is stable by finite intersections and countable unions.
We still denote $\mathcal{U}_{\mathbb{N}}$ the ordered family of subspaces of $X^{\mathbb{N}}$
which are countable unions of finite intersections
of subspaces of $X^{\mathbb{N}}$ of the form

$$P^U_{\geq q}(X^{\mathbb{N}}) \quad \text{or} \quad P^U_{\leq q}(X^{\mathbb{N}}) \quad \text{with} \quad U \in \mathcal{U} \text{ and } q \in Q\,.$$

Here, Q is a countable and dense subset of $[0,1]$,
stable by the relation $q_1 + q_2 = q_3 + q_4$ of $[0,1]^4$.

Corollary. – *Let $\mathcal{E}_{\mathbb{N}}$ be the localic topos of sheaves on the site*

$$(\mathcal{U}_{\mathbb{N}}, J_{\mathbb{N}})$$

*consisting in the ordered set $\mathcal{U}_{\mathbb{N}}$ endowed with the topology $J_{\mathbb{N}}$.
Then we have a triple equivalence*

$$\mu \longleftrightarrow J_\mu \longleftrightarrow p_\mu$$

between

- *probability measures μ on $\mathcal{U}$,*
- *subtoposes $\mathcal{E}_\mu = \widehat{(\mathcal{U}_{\mathbb{N}})}_{J_\mu}$ of $\mathcal{E}_{\mathbb{N}}$ which are "two-valued",*
- *points p_μ of the topos $\mathcal{E}_{\mathbb{N}}$.*

Explicitation of the topology which defines the topos of measures :

The localic topos of probability measures on $\mathcal{U}$

$$\mathcal{E}_{\mathbb{N}} = (\mathcal{U}_{\mathbb{N}})_{J_{\mathbb{N}}}$$

is defined as the topos of sheaves on $\mathcal{U}_{\mathbb{N}}$
for the topology $J_{\mathbb{N}}$ which was introduced as a generated topology.
Here is a characterization of this topology :

Lemma. – *A family of morphisms $(P_i \hookrightarrow P)_{i \in I}$ of $\mathcal{U}_{\mathbb{N}}$
is $J_{\mathbb{N}}$-covering if and only if it contains
a countable subfamily $(P_{i_n})_{n \in \mathbb{N}}$ such that the difference*

$$P - \bigcup_{n \in \mathbb{N}} P_{i_n}$$

*is "negligible" in the sense that it is contained
in a countable union of subspaces of the form*

- $\{x_\bullet \in X^{\mathbb{N}} \mid p_-^U(x_\bullet) < p_+^U(x_\bullet)\}$ *with* $U \in \mathcal{U}$,
- $\{x_\bullet \in X^{\mathbb{N}} \mid \sup\limits_{n \in \mathbb{N}} p_-^{U_n}(x_\bullet) < p_-^U(x_\bullet)\}$

for an increasing sequence of subsets $U_n \in \mathcal{U}$, $n \in \mathbb{N}$, with $U = \bigcup\limits_{n \in \mathbb{N}} U_n$.

Non-triviality of the topos of probability measures :

We remark :

Corollary. – *If $\mathcal{U}$ is a family of subsets of a non-empty set X which is stable by finite intersections and countable unions, we have :*

(i) *Any element $x \in X$ defines a probability measure δ_x on $\mathcal{U}$ by*

$$\mathcal{U} \ni U \longmapsto \begin{cases} 1 & \text{if } x \in U \,, \\ 0 & \text{if } x \notin U \,. \end{cases}$$

(ii) *A fortiori, the localic topos of probability measures on $\mathcal{U}$*

$$\mathcal{E}_{\mathbb{N}} = \widehat{(\mathcal{U}_{\mathbb{N}})}_{J_{\mathbb{N}}}$$

always has points associated with elements $x \in X$, and the full space

$$X^{\mathbb{N}}$$

is never negligible for the topology $J_{\mathbb{N}}$.