Geometry according to Grothendieck: glimpses from “Récoltes et Semailles”

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One of the biggest losses in the History of Mathematics:

- In 1970, at age 42,
  Grothendieck resigned from IHES.
- In 1973, he became a professor in Montpellier,
  far away from his previous scientific environnement,
  after unsuccessfully applying for positions at
  - Collège de France,
  - CNRS,
  - Paris XI University (in Orsay).
- In 1984, he stopped teaching
  and eventually got a CNRS position.
- In 1988, at age 60,
  he officially retired.
- He died in 2014, at age 86, after 25 years in isolation.

Our very partial knowledge of Grothendieck's mathematics after the 1970's:

- "Topological algebra":
  - $\longrightarrow$ "Pursuing stacks" (letter to Quillen)
  - $\longrightarrow$ "Les dérivateurs" (Derivators)
  - $\longrightarrow$ "Esquisse d'un programme"
  - $\longrightarrow$ Letters to Thomason, Breen, ...
- "Anabelian geometry",
  (Grothendieck-)Teichmüller tower,
  "dessins d'enfants" (children's drawings):
  - $\longrightarrow$ "Esquisse d'un programme"
  - $\longrightarrow$ Letter to Faltings
  - $\longrightarrow$ "La longue marche à travers la théorie de Galois"

- “Tame topology” (“topologie modérée”):

  → Only a few pages in
  “Esquisse d’un programme”.

  → We will never know for sure whether the present development of
  “*o*-minimal structures” theory realizes Grothendieck’s vision.

- “Scheme-theoretic point of view on
regular polyhedra and regular configurations”:

  → Only a few paragraphs in
  “Esquisse d’un programme”.

  → According to Grothendieck, this is
  the most modest of his 12 “master themes”,
  but “even a life-time’s work would not exhaust it” . . .

- More to be discovered in his 70 000 pages Lasserre papers ?

At least, do we know well Grothendieck's algebraic geometry?

Of course, we have thousands of pages:

- His programmatic 1958 ICM talk:
  "The cohomology theory of abstract algebraic varieties".

- Grothendieck's writing of his seminar talks:

  $\longrightarrow$ Bourbaki seminar
  (in particular his FGA talks on
  "construction techniques" in algebraic geometry)

  $\longrightarrow$ Cartan seminar
  (in particular his talks on construction techniques in analytic geometry
  and the Teichmüller tower)

  $\longrightarrow$ Chevalley seminar
  (for instance his talk on intersection theory)

- Some very important papers:

  - $\longrightarrow$ “Sur quelques points d’algèbre homologique” (Tohoku)
  - $\longrightarrow$ “On the De Rham cohomology of algebraic varieties”
  - $\longrightarrow$ “Crystals and the De Rham cohomology of schemes”
  - $\longrightarrow$ “Standard conjectures on algebraic cycles”
    ($=$ his only published text on motives)

- His letters:

  - $\longrightarrow$ Published correspondence with Serre.
  - $\longrightarrow$ Many letters to other mathematicians.
    Ex: famous letter to Tate.
  - $\longrightarrow$ Question: how many unpublished letters?

    (NB: According to the Montpellier website,
    10 000 of the 28 000 pages Grothendieck archives
    could not be made public
    without other people’s permission.)

- The EGA volumes I, II, III, IV:
  - $\longrightarrow$ The introduction to EGA I mentioned 13 chapters to be written.
  - $\longrightarrow$ SGA was meant to be a preparation for the future writing of more advanced new EGA chapters.
  - $\longrightarrow$ There exist “pre-notes” for EGA V on “hyperplane sections and conic projections”.
- The SGA lecture notes:

  From SGA 1
  to SGA 7.

Some statements of “Récoltes et Semailles”:

- Already in 1960’s, he was seeing
  “a proud bunch of books that were crying out to be written”
  (“un fier paquet de livres qui réclamaient à cor et à cri d’être écrits”).

  $\longrightarrow$ He “would have been willing to write them himself”
  but “he couldn’t do everything at the same time.”

  $\longrightarrow$ As he writes RS ($=$ 20 years later),
  “none of these books has been written, to his knowledge”,
  “except for the PhD theses written under his guidance”.

- Even excluding the SGA talks, he gave multiple series of talks at IHES.

  → Presumably, he always had written notes
  (as he always thought by writing).

  → Some series of talks have been formally published.

  Ex: "Crystals and the De Rham cohomology of schemes".

  → Some other series of talks were never published.

  Ex: a series of IHES talks on motives.

  Question: What did the notes become ?

  Are these the notes on "motifs"
  that are part of
  the Montpellier archives ?

- Most SGA talks were given by him.

  $\longrightarrow$ He entrusted his written notes to volunteers,
  who were responsible for completing them
  with all the necessary details.

  $\longrightarrow$ Some (or many?) of these completed texts
  have been published in the SGA series
  without his name appearing as author.

  $\longrightarrow$ The notes of several Grothendieck SGA talks
  (especially in SGA 5)
  have never been completed and never published.

  Ex: the first talk of SGA 5
  (an overview of duality in multiple types of geometric contexts
  $=$ ubiquity of the "six operations" formalism),
  Ex: the last concluding talk of SGA 5
  (an overview of open questions).

- He is extremely unhappy
  with the SGA 5 volume
  that was eventually published.

  $\longrightarrow$ According to his words,
  SGA 5 ($+$ SGA $4^{1/2}$)
  $=$ “édition-massacre” (massacre edition).

  $\longrightarrow$ This is especially pityfull
  as, for him,
  the SGA 4-SGA 5 seminars
  are the master part (“partie maîtresse”)
  of his work in geometry.

- Why is he so unhappy?
  In particular because
  vision and perspective (which he outlined at length in his talks)
  have disappeared.

  Ex: The expression “six operations formalism” is not in SGA 5.
  We would not know it without RS.

A parenthesis of wishful dreaming:

- Is it possible to find in IHES archives
  a list of all talks
  (with titles and possibly abstracts)
  that Grothendieck gave there?
- Is it possible to find
  his written notes for at least
  part of these talks?

$\longrightarrow$ Even for the notes
which were eventually completed by others,
his original notes may be interesting.

A partial replacement for Grothendieck's written notes: the mathematical contents of RS

Indeed,
what he mentions in RS
is what is most important for him:

- The main simple unifying lines of his vision.
- The basic key notions.
- The basic (or sometimes more elaborate) key facts.

The three “aspects” of things that can be thought of mathematically:

- “le nombre” (number)
  $\longrightarrow$ “arithmetic” aspect,
- “la grandeur” ($\sim$ quantity, magnitude)
  $\longrightarrow$ “metric” or “analytical” aspect,
- “la forme” (form)
  $\longrightarrow$ “geometric” aspect.

What fascinates him most:

Neither number (“le nombre”)
nor quantity (“la grandeur”)
but <u>form</u> (“la forme”)
and, among its thousand faces,
<u>structure</u>.

The good attitude to study structures:

- Structures cannot be invented:

  they are already there,

  waiting for us to

  discover them,
  grasp and understand them ("l'appréhension"),
  express them.

- To uncover and express structures,

  we have to

  invent and refine languages,

  build theories

  that are meant to account for

  what has been seen and understood.

- The quality of our inventiveness and imagination

  is the quality of the attention we pay,
  of our listening.

Grothendieck's definition of geometry (or of his "new geometry"):

- Arithmetic
  = science of discrete structures.
- Analysis
  = science of continuous structures.
- Geometry
  straddles ("est à cheval entre")
  these two types of structures.

$\longrightarrow$ His "new geometry" is a synthesis between

- the arithmetic world of
  "spaces without continuity principle",
- the world of continuous quantity
  ("le monde de la grandeur continue").

$\longrightarrow$ "In the new vision,
these two once separated worlds become one."

A characteristic feature of geometry:

- His feeling in 1955,

  when he moved from analysis to geometry:

  > “I still remember this striking impression,
  > as if I were leaving arid steppes,
  > to find myself suddenly in a sort of
  > “promised land” with luxuriant riches,
  > multiplying infinitely
  > whenever the hand can reach, to pick or to dig . . .”

$\longrightarrow$ As opposed to analysis,
geometry $=$ land of infinite luxuriant riches.

$\longrightarrow$ As opposed to arithmetic (?),
geometry $=$ land where luxuriant riches
readily multiply infinitely
under our hand.

The continuous and the discrete in RS:

The adjective “continuous” (“continu”),
and derived words (as the verb “to continue”)
appear hundreds of times in RS
with two different meanings:

- mathematical meaning
  (to which we are going to come back),
- ordinary language meaning,
  applied to the research work.

$\longrightarrow$ Continuity
is what we can give (and have to give)
in our work.

$\longrightarrow$ Discontinuity
(taking the form of unexpected discoveries)
is what can be given to us.

Which type of continuity in our work?

- Continuity of effort:

  never stop too early,
  never stop looking for a deeper understanding
  if a "feeling of unease" ("un sentiment de malaise")
  is still there.

- Always choose to express and develop first
  what is immediately in front of our eyes.

- Continuation (almost as analytic continuation)
  of what has already been developed.

Counter-example:

- He would have expected the SGA series to continue after him:

  "The SGA 7 operation is not a continuation,
  but I feel it as a kind of brutal "saw cut" (or chainsaw cut),
  bringing the SGA series to an end."

Which type of discontinuity in discoveries?

- **First example:**
  the discovery that
  the formalism of the six operations and biduality
  applies both to continuous coefficients and discrete coefficients.

  "This ubiquity appeared at Spring 1963
  as a surprise almost impossible to believe
  (une surprise à peine croyable)."

- **Second example:**
  the discovery that
  the cohomology of the crystalline site,
  with coefficients in the structure sheaf,

  identifies with De Rham cohomology.

  "A relation that I discovered in 1966
  and which stunned me at the time
  (qui m'a alors sidéré) . . ."

Grothendieck's new geometry as a world which unifies the continuous and the discrete:

"We can consider that the new geometry is, above all,
a synthesis between these two worlds,
which until then had been adjoining
and closely interdependent, but yet separate:
the "arithmetic" world,
in which "spaces" without a principle of continuity live,
and the world of continuous quantity.

In the new vision,
these two formerly separate worlds
become one."

First type of synthesis: the building of a common house

- The common house of the continuous

  and the discrete

  is: Topos theory.

- Based on a double discovery:

  $\longrightarrow$ All categories

  $$\mathrm{Sh}(X) = \{\text{sheaves on a topological space } X\}$$

  and

  $$\widehat{\mathcal{C}} = \{\text{presheaves on a small category } \mathcal{C}\text{, e.g. a group or a monoid}\}$$

  share the same general properties.

  $\longrightarrow$ The construction

  $$\begin{array}{ccc} X & \longmapsto & \mathrm{Sh}(X) \\ (f : X \to Y) & \longmapsto & (\mathrm{Sh}(Y) \xrightarrow{f^*} \mathrm{Sh}(X), \mathrm{Sh}(X) \xrightarrow{f_*} \mathrm{Sh}(Y)) \end{array}$$

  is an embedding of

  the world of continuity (topological spaces)

  into

  the world of discrete structures (categories).

Second type of synthesis: “ubiquity”

This is the discovery that some phenomena
are “ubiquitous” both with “continuous coefficients”
and with “discrete coefficents”.

- First most important example: duality
  $=$ a network of linear functorial constructions and their properties
  which jointly generalises $\begin{cases}\text{Poincaré duality for singular homology,}\\ \text{Serre duality for coherent Modules,}\end{cases}$
  and appear “ubiquitous” in the most diverse topological
  or geometric contexts,
  $=$ system of
  $\begin{cases} - & \text{six operations formalism } (\otimes, \mathrm{Hom}, f^*, f_*, f_!, f^!), \\ - & \text{notion of constructible objects,} \\ - & \text{biduality.} \end{cases}$
- Second example:
  (Grothendieck-) Riemann-Roch for $\begin{cases}\text{continuous coefficients,}\\ \text{discrete coefficients.}\end{cases}$

Third type of synthesis: bridges between the discrete and the continuous

Remarks:

- The word “bridge” has been chosen by O.Caramello for her theory of “topos-theoretic bridges” which is a general theory of relations between the contents of different mathematical theories.
- In RS Grothendieck uses the word “bridge” (pont) a few times:

  This is always about
  the work of Zoghman Mebkhout
  (especially the “théorème du bon Dieu”
  $=$ Riemann-Hilbert correspondence)
  which, according to him, realizes

  “a bridge between topology, algebra and analysis”.

Two types of bridges between the discrete and the continuous:

- Bridges that consist in relations or equivalences:
  - $\longrightarrow$ First embryo: the Weil conjectures which predict a tight relationship between arithmetic and topology.
  - $\longrightarrow$ The new theories and developments which the Weil conjectures inspired to Grothendieck:
    - — construction of cohomology invariants that interplay topology and arithmetic,
    - — trace formulas for these invariants,
    - — the "yoga of weights".
  - $\longrightarrow$ The Riemann-Hilbert correspondence as an equivalence between (linear) derived categories of differential nature (systems of linear PDE's) and of discrete nature.
  - $\longrightarrow$ The "anabelian geometry" program, based on the key fact that objects of purely arithmetic nature $(\mathrm{Gal}(\mathbb{Q}) = \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}))$ naturally act on objects of purely topological nature.

- Bridges that consist in objects or theories
  that can specialize or localize
  both in the direction of the discrete
  and the direction of the continuous:

  $\longrightarrow$ Scheme theory over Spec$(\mathbb{Z})$ with its natural base change functors to

  $$\begin{cases} - & \text{discrete geometry over finite fields,} \\ - & \text{continuous geometry over } \mathbb{R} \text{ or } \mathbb{C}. \end{cases}$$

  $\longrightarrow$ The theory of sites $(\mathcal{C}, J)$
  ($\mathcal{C}$ = small category, J = Grothendieck topology on $\mathcal{C}$)

  which specializes to both $\begin{cases} \text{(continuous) topological spaces,} \\ \text{(discrete) small categories,} \end{cases}$

  and allows definitions of intermediate nature such as
  étale sites, crystalline sites, ...

  $\longrightarrow$ The conjectural theory of motives that should specialize both to

  $$\begin{cases} - & \text{cohomology theories with discrete coefficients} \\ & \text{such as } \ell\text{-adic or singular cohomology,} \\ - & \text{Hodge cohomology which is subject} \\ & \text{to continuous variations.} \end{cases}$$

The home of Grothendieck's new geometry: toposes

"The name 'topos' was chosen to suggest that
it is the ultimate object (l'objet par excellence)
to which the topological intuition applies.

By the rich cloud of mental images that this name elicits,
it should be considered
as being the equivalent of the term 'space',
with a greater emphasis on the 'topological' specificity of the notion."

- Why "ultimate" (objet "par excellence")?

  $\longrightarrow$ The notion of topos is more general
  than any previous notion of space.

  $\longrightarrow$ It is not too general:

  All familiar most essential geometric intuitions and constructions
  can be more or less easily transposed to the new notion of topos.

$\longrightarrow$ It is at the same time general enough
to encompass a host of situations which,
until then, were not considered to give rise
to intuitions of a ‘topological-geometric’ nature.

- The qualification “par excellence” (ultimate)
is not only a feeling,
it is justified by the double definition of toposes:

$\longrightarrow$ Constructive definition

as categories that can be (geometrically) presented
as categories of sheaves on some site $(\mathcal{C}, J)$.

$\longrightarrow$ Axiomatic characterization (Giraud’s theorem)

as categories which share with

Set $(=$ category of sets and maps)

the same list of constructive categorical properties.

Toposes as "pastiches" of Set, sharing the same list of "strongly typed properties":

- Any mathematical theory $\mathbb{T}$ has models in an arbitrary topos $\mathcal{E}$, just as it has set-theoretic models.
- If $\mathbb{T}$ is first-order, these models make up a category $\mathbb{T}$-Mod$(\mathcal{E})$.
- If $\mathbb{T}$ is "first-order geometric"
(i.e. uses constructions only in terms of colimits and finite limits),
any morphism of toposes

$$(\mathcal{E} \xrightarrow{f} \mathcal{S}) = (\mathcal{S} \xrightarrow{f^*} \mathcal{E}, \mathcal{E} \xrightarrow{f_*} \mathcal{S})$$

induces a pull-back functor

$$f^* : \mathbb{T}\text{-Mod}(\mathcal{S}) \longrightarrow \mathbb{T}\text{-Mod}(\mathcal{E})\,.$$

- If $\mathbb{T}$ is furthermore algebraic (or more generally cartesian),
such a topos morphism $f : \mathcal{E} \to \mathcal{S}$
also induces a push-forward functor

$$f_* : \mathbb{T}\text{-Mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-Mod}(\mathcal{S})$$

right-adjoint to f^*.

Toposes as the natural context for cohomology:

- If $(\mathcal{E}, \mathcal{O})$ = "ringed topos"
$$\begin{array}{c} \| \\ \text{topos } \mathcal{E} + \text{inner ring } \mathcal{O} \\ \qquad\qquad (= \text{sheaf of rings}), \end{array}$$
  the associated abelian category
$$\begin{array}{rl} \mathrm{Mod}_{\mathcal{O}} = & \{\text{inner } \mathcal{O}\text{-Modules in } \mathcal{E}\} \\ & \| \\ & \text{sheaves of } \mathcal{O}\text{-modules} \end{array}$$
  has all the "Tohoku" properties which are needed to develop cohomology.

- If $(\mathcal{E}, \mathcal{O}) \to (\mathcal{S}, \mathcal{A})$ = morphism of ringed toposes
$$= \begin{cases} \mathcal{E} \xrightarrow{f} \mathcal{S}\,, \\ f^*\mathcal{A} \to \mathcal{O} \quad \text{or} \quad \mathcal{A} \to f_*\mathcal{O}\,, \end{cases}$$
  there is an associated pair of adjoint linear functors
$$(f^* : \mathrm{Mod}_{\mathcal{A}} \longrightarrow \mathrm{Mod}_{\mathcal{O}}\,, \quad f_* : \mathrm{Mod}_{\mathcal{O}} \longrightarrow \mathrm{Mod}_{\mathcal{A}})$$
  and associated derived functors
$$Lf^*\,, \quad Rf_*$$
  between derived categories.

Toposes as a context for duality:

- If $G =$ object of a topos $\mathcal{E}$, the functor
$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}\,, \\ F & \longmapsto & F \times G \end{array}$$
has a right adjoint
$$\begin{array}{ccc} D & \longmapsto & D^G = \mathcal{H}om(G,D) \\ & & \| \\ & & \text{sheaf of sheaves morphisms from } G \text{ to } D. \end{array}$$
There is a natural transformation
$$G \longrightarrow \mathcal{H}om(\mathcal{H}om(G,D),D)\,.$$
- If $(\mathcal{E},\mathcal{O}) =$ commutative ringed topos,
any $\mathcal{O}$-Module N defines a tensor functor
$$\begin{array}{ccc} \mathrm{Mod}_{\mathcal{O}} & \longrightarrow & \mathrm{Mod}_{\mathcal{O}}\,, \\ M & \longmapsto & M \otimes_{\mathcal{O}} N \end{array}$$
left adjoint to a functor
$$\begin{array}{ccc} D & \longrightarrow & D^N = \mathcal{H}om_{\mathcal{O}}(N,D) \\ & & \| \\ & & \text{sheaf of } \mathcal{O}\text{-linear sheaves morphisms.} \end{array}$$
There is a natural transformation
$$N \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{H}om_{\mathcal{O}}(N,D),D)\,.$$

Remark:

$N \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{H}om_{\mathcal{O}}(N, D), D)$

is an isomorphism if

$\mathcal{E} = \text{Set}$, $\mathcal{O} = \text{field } k$, $\dim_k(D) = 1$

and N has finite dimension.

Toposes as a context for biduality:

- If $(\mathcal{E}, \mathcal{O}) =$ commutative ringed topos,

$$\bigotimes_{\mathcal{O}}, \quad \mathcal{H}om_{\mathcal{O}}$$

have associated derived functors

$$\overset{L}{\bigotimes}_{\mathcal{O}}, \quad R\mathcal{H}om_{\mathcal{O}}.$$

- If $D =$ object of the derived category of $\mathrm{Mod}_{\mathcal{O}}$ verifying some hypotheses, there is a natural transformation

$$N \longrightarrow R\mathcal{H}om_{\mathcal{O}}(R\mathcal{H}om_{\mathcal{O}}(N, D), D).$$

Grothendieck biduality question:

Does there exist D (= "dualizing complex")
such that it is an isomorphism for
general enough objects N in the derived category of $\mathrm{Mod}_{\mathcal{O}}$?

$\longrightarrow$ It works or should work for all natural geometric contexts.

$\longrightarrow$ It cannot work without derived categories.

Toposes as a “powerful tool for discovery” (“puissant outil de découverte”):

- From Grothendieck’s own words:

“The two consecutive seminars SGA 4 and SGA 5
(which for me are like a single seminar)
develop from nothing

both the powerful instrument of synthesis and discovery
that the language of toposes represent

and the perfectly developed and perfectly efficient tool
that is étale cohomology,

better understood in its essential formal properties,
from then on,
than even the cohomological theory of ordinary spaces”.

Grothendieck topos-oriented principles:

- Start from a (possibly elementary) notion or fact
  we already know in the context of

  some classical geometric theory
  (such as continuous topological spaces)

  or even only over a point
  (remembering that Set $=$ topos of sheaves on a point).

  Ex: start from the basic fact that
  a finite dimensional vector space
  is the dual of its dual.

- Look for a new formulation
  that makes sense for arbitrary toposes or morphisms of toposes
  (possibly with extra structure such as inner rings)
  and could be true
  for very different families of toposes and their morphisms
  stemming out from diverse natural domains of mathematics.

What can be discovered by respecting these principles:

- The classical starting notion or fact
  can in this way
  be transported and transformed
  into completely new and possibly much deeper
  notions or facts
  in a multitude of different contexts.

  Ex: the metamorphosis
  of the phenomenon of biduality of finite dimensional vector spaces
  into biduality for complexes of linear sheaves which

  - requires to work at the level of derived categories,
  - leads to the notion of dualizing complex
    (generalizing the 1-dimensional vector space),
  - leads to the notion of "constructible complex"
    (generalizing the property of finite dimension),
  - is usually very deep,
  - is "ubiquitous" in geometry, as Grothendieck realized.

Toposes as a broad enough notion to classify any type of mathematical structure:

"Other notable examples of toposes
that are not ordinary spaces,
and for which there seems to be
no satisfactory substitute in terms of 'accepted' notions,
are 'classifying' toposes
for just about any kind of mathematical structure
(at least those 'expressed in terms of finite limits and arbitrary colimits')."

Classifying toposes for fist-order ‘geometric’ theories:

$\mathbb{T}$ = first-order theory
which is ‘geometric’
(i.e. whose semantic interpretation
as ‘models’ in toposes
only makes use of colimits and finite limits).

Theorem (Lawvere, Joyal, Makkai, Reyes, ...
building on first examples given in
Monique Hakim’s PhD thesis under Grothendieck):

The functor

$$\begin{array}{rcl} \mathcal{E} = \text{topos} & \longmapsto & \mathbb{T}\text{-mod}(\mathcal{E})\,, \\ (\mathcal{E} \xrightarrow{f} \mathcal{S}) & \longmapsto & (f^* : \mathbb{T}\text{-mod}(\mathcal{S}) \to \mathbb{T}\text{-mod}(\mathcal{E})) \end{array}$$

is representable by a (unique up to equivalence)
‘classifying topos’ $\mathcal{E}_{\mathbb{T}}$
characterized by natural equivalences

$$\{\textit{topos morphisms}\ \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\} \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathcal{E})\,.$$

Classifying toposes for higher-order structures:

"When we consider some type of variety structure
(which may be topological, differential, real or complex analytic, Nash, etc.)
or even smooth schematic on a given basis),
we find in each case a particularly attractive topos,
which deserves the name of universal variety (of the species considered).

Its homotopic invariants
(and in particular its cohomology,
which deserves the name of 'classifying cohomology'
for the species of variety under consideration)
should have been studied and known for a long time."

Explicitation of what Grothendieck means:

- First ingredient: a category of parametrizing varieties

$\mathcal{C}$ = category of topological varieties
differential varieties,
real analytic varieties,
complex analytic varieties,
Nash varieties,
schemes finitely presented over $\mathbb{Z}, \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \ldots$

J = Grothendieck topology on $\mathcal{C}$
(ordinary, étale, smooth, fpqc ...)
which is subcanonical, meaning

$$\ell : \mathcal{C} \xrightarrow{\text{Yoneda}} \widehat{\mathcal{C}} \xrightarrow{\text{sheafification}} \widehat{\mathcal{C}}_J = \mathcal{S}$$

is a fully faithful embedding

$$\mathcal{C} \hookrightarrow \mathcal{S} = \widehat{\mathcal{C}}_J = \text{topos of sheaves on } (\mathcal{C}, J)\,.$$

- Second ingredient: a natural fibered category of relative geometric structures

$$\begin{array}{ccl} S & \longmapsto & \text{groupoid } \mathcal{M}(S), (= \text{objects} + \text{isomorphisms}) \\ \| & & \text{of relative} \\ \text{object of } \mathcal{C} & & \text{geometric structures over } S, \\ & & \\ (S' \to S) & \longmapsto & \text{functor } \mathcal{M}(S) \to \mathcal{M}(S'). \\ \| & & \\ \text{morphism of } \mathcal{C} & & \end{array}$$

Ex: $S \mapsto \mathcal{M}(S) =$ groupoid of morphisms of $\mathcal{C}$

$$p : X \longrightarrow S$$

- which are proper (= relatively compact)
  smooth
- whose fibers have fixed discrete invariants (ex: the genus g for relative curves),
- possibly with some extra structure (ex: structure of abelian variety, relative normal crossings divisor, embedding in a fixed projective space, polarisation, ...).

$$(S' \to S) \longmapsto [(X \xrightarrow{p} S) \mapsto (X \times_S S' \to S')] .$$

- Most important particular case:

$$\begin{array}{ccc} S & \longmapsto & \mathcal{M}_{g,n}(S) \\ \| & & \| \\ \text{finitely presentable} & & \text{groupoid of} \\ \text{scheme over } \mathbb{Z} & & \text{relative smooth projective curves} \\ & & p : X \to S \\ & & \text{of genus } g, \\ & & \text{with } n \text{ points.} \end{array}$$

- Other natural examples:

Ex: If $G =$ group bundle
over a proper (compact) object C of $\mathcal{C}$,

$$S \longmapsto \mathcal{M}(S) = \text{ groupoid of } G\text{-torsors over } S \times C.$$

Ex: If $C =$ proper object of $\mathcal{C}$,

$$S \longmapsto \mathcal{M}(S) = \text{ groupoid of } \begin{cases} \text{“local systems” of rk } r, \\ \ell\text{-adic sheaves of rk } r, \end{cases}$$

over $S \times C$.

What may happen:

- The groupoids $\mathcal{M}(S)$ may be sets (no automorphisms) or not.
  If they are sets, $\mathcal{M}$ is a presheaf on $\mathcal{C}$.
  This presheaf may be a J-sheaf or not.
  If it is a J-sheaf, it may be representable by an object of $\mathcal{C}$
  (called the moduli variety of $\mathcal{M}$) or not.
- Anyway, if $\mathcal{M}$ is a presheaf, it may be sheafified by

$$\begin{array}{ccc} \mathcal{M} & \longmapsto & \widetilde{\mathcal{M}} \\ \text{in } \widehat{\mathcal{C}} & & \text{in } \widehat{\mathcal{C}}_J = \mathcal{S} \end{array}$$

  and the relative category

$$\mathcal{E}_{\mathcal{M}} = \mathcal{S}/\widetilde{\mathcal{M}}$$

  is a topos, with associated homotopy (and cohomology) invariants.
  If $\mathcal{M}$ is representable by an object $S_{\mathcal{M}}$ of $\mathcal{C}$,
  this topos and its invariants
  are the same as those of $S_{\mathcal{M}}$.

The general case:

- In general, the groupoids $\mathcal{M}(S)$ are not sets and $\mathcal{M}$ is just a fibered category.
- Most often (but not always), this fibered category is a J-stack (champ):
  - isomorphisms may be defined J-locally by gluing,
  - objects may be constructed J-locally through "descent data".
- Even if $\mathcal{M}$ is not a J-stack, it has an associated stack (just as any presheaf has an associated sheaf).
- If $\mathcal{M}$ is a stack, it may or may not be a multiplicity (multiplicité) J-locally isomorphic to objects of $\mathcal{C}$.
- In all cases, $\mathcal{M}$ and its stackification $\widetilde{\mathcal{M}}$ define a relative topos

$$\mathcal{E}_{\mathcal{M}} \longrightarrow \mathcal{S} = \widehat{\mathcal{C}}_J$$

which has homotopy (and cohomology) invariants.

Most important case:

(Deligne-Mumford building on Riemann, Teichmüller, Grothendieck)

- $\mathcal{M}_{g,n} =$ multiplicity over $\mathrm{Spec}(\mathbb{Z})$
  of relative dimension $3g - 3 + n$
  (if $g \geq 2$
  or $g = 1$, $n \geq 1$
  or $g = 0$, $n \geq 3$).
- Each $\mathcal{M}_{g,n}$ has a modular compactification

$$\mathcal{M}_{g,n} \xhookrightarrow{\text{open}} \overline{\mathcal{M}_{g,n}}$$

$\|$

proper multiplicity over $\mathrm{Spec}(\mathbb{Z})$
defined by a broader classification problem.

- The boundary strata of $\overline{\mathcal{M}_{g,n}}$
  are constructed from some
  $\mathcal{M}_{g',n'}$ of lesser dimensions.

An implicit question:

Consider as before a natural "geometric" fibered category:

$$\begin{array}{ccc} S & \longmapsto & \mathcal{M}(S)\,. \\ \text{object of a category } \mathcal{C} & & \\ \text{endowed with a topology } J & & \end{array}$$

Then, the fibered category $\mathcal{M}$ has a "classifying topos" $\mathcal{E}_{\mathcal{M}}$ over the base topos $\mathcal{S} = \widehat{\mathcal{C}}_J$.

Question: *How to interpret topos morphisms*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{E}_{\mathcal{M}} \arrow[d] \\ & \mathcal{S} = \widehat{\mathcal{C}}_J \end{tikzcd} \quad ?$$

What is a model of a second-order theory in a relative topos $\mathcal{E} \to \mathcal{S}$?

- $\longrightarrow$ Develop a theory of classifying relative toposes for (to be defined) "geometric" second-order theories.
- $\longrightarrow$ Generalize to higher-order theories.

The ubiquity of the "six operations formalism" for duality with continuous and discrete coefficients:

- Discovered by Grothendieck between 1956 and 1963
  - first for complexes of $\mathcal{O}_X$-Modules with coherent cohomology over nœtherian schemes X (understanding and generalizing Serre duality),
  - secondly for discrete coefficients in the context of étale cohomology of schemes (understanding and generalizing Poincaré duality in a technically much more difficult situation than ordinary cohomology).
- "These two extreme cases were sufficient to found the conviction of the <u>ubiquity</u> of this formalism in <u>all geometric situations</u> giving rise to a Poincaré type <u>duality</u>."

- Grothendieck convinced himself that the
“catch-all” formalism of the “six operations”
works or should work
for the “most important types of ‘spaces’ that have been introduced so far in geometry:

  ‘algebraic spaces’ (such as schemes, scheme multiplicities, etc.),
  analytic spaces (both complex analytic, and rigid-analytic and similar),
  topological spaces
  (while waiting for the context of ‘tame spaces’ of all kinds),
  and surely many others,
  such as the category Cat of small categories, serving as homotopy types.”
- He mentions that he had fully checked
some cases as exercises,
so as for instance <u>the case of piecewise-linear spaces</u>.

  $\longrightarrow$ Never published ...
- He mentions that he had given a SGA 5 talk about
  - the general formalism of the “six operations”,
  - the many geometric situations where it applies.

  $\longrightarrow$ The published SGA 5 volume doesn’t contain this talk.

The context of the "six operations formalism":

- A "geometric" category

$$\mathcal{C}$$

endowed with a functor

$$\begin{array}{rcl} \mathcal{C} & \longrightarrow & \text{2-category of (commutative) ringed toposes,} \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X)\,. \end{array}$$

$\rightarrow$ "Continuous coefficients"
means that $\mathcal{O}_X$ is the "structure sheaf"
(consisting of algebraic, analytic, differential, ... functions on X)
that is part of the definition of X as a geometric object.

$\rightarrow$ "Discrete coefficients"
means that $\mathcal{O}_X$ is the "constant sheaf"
consisting of "locally constant functions"
with values in some fixed ring $\wedge$ $(= \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{R}, \ldots)$.

- The associated linear categories

$$X \longmapsto \begin{cases} \mathrm{Mod}_{\mathcal{O}_X} = \text{abelian category of } \mathcal{O}_X\text{-Modules in } \mathcal{E}_X, \\ D_X = \text{associated derived category.} \end{cases}$$

The four operations that are already given:

- For any object X of the geometric category $\mathcal{C}$:

$$(\otimes_{\mathcal{O}_X}, \mathcal{H}om_{\mathcal{O}_X})$$

  and the associated derived functors

$$(\overset{L}{\otimes}_{\mathcal{O}_X}, R\mathcal{H}om_{\mathcal{O}_X}) \, .$$

- For any morphism $f : X \to S$ of $\mathcal{C}$:

$$(\mathrm{Mod}_{\mathcal{O}_S} \xrightarrow{f^*} \mathrm{Mod}_{\mathcal{O}_X}, \mathrm{Mod}_{\mathcal{O}_X} \xrightarrow{f_*} \mathrm{Mod}_{\mathcal{O}_S})$$

  and the associated derived functors

$$(Lf^*, Rf_*) \, .$$

Remark:
Lf^* is left-adjoint to Rf_*
if f^* or f_* has finite cohomological dimension.

The two extra operations realizing duality:

- In most (all ?) situations which are significant,
  any morphism $f : X \to S$ of $\mathcal{C}$
  induces a pair of adjoint functors

  $$(D_X \xrightarrow{f_!} D_S, D_S \xrightarrow{f^!} D_X)$$

  such that
  - $(f \mapsto f_!) =$ covariant functor,
  - $(f \mapsto f^!) =$ contravariant functor,
  - $f_! = Rf_*$ if $(f : X \to S)$ is "proper"
    ($=$ relatively compact in a sense to be defined in the context of $\mathcal{C}$),
  - $f^! = f^*$ (and so: $f_! =$ "extension by 0", left-adjoint to $f^* =$ restriction)
    if f is an open embedding (or étale) morphism
    (which can be seen at the level of $\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_S$).

The duality diagram:

- For any morphism $X \xrightarrow{f} S$ of $\mathcal{C}$,
  and any object $\mathcal{M}_S$ of D_S,
  the square

$$\begin{tikzcd} D_X \arrow[r, "R\mathrm{Hom}(\bullet, f^! \mathcal{M}_S)"] \arrow[d, "f_!"'] & D_X \arrow[d, "Rf_*"] \\ D_S \arrow[r, "R\mathrm{Hom}(\bullet, \mathcal{M}_S)"] & D_S \end{tikzcd}$$

  should be commutative (up to canonical isomorphism).

Constructibility and biduality:

- There should be a notion of
  "constructible objects of D_X", for any object X of $\mathcal{C}$,
  such that
  - the structure sheaf $\mathcal{O}_X$ is constructible,
  - constructible objects are stable
    under all six operations,
  - any X has a "dualizing complex" $\mathcal{D}_X$ in D_X
    realizing biduality

    $$\mathcal{M}_X \xrightarrow{\sim} R\mathcal{H}om(R\mathcal{H}om(\mathcal{M}_X, \mathcal{D}_X), \mathcal{D}_X)$$

    for any <u>constructible</u> object $\mathcal{M}_X$ of D_X,
    and verifying

    $$\mathcal{D}_X \xrightarrow{\sim} f^! \mathcal{D}_S$$

    for any $f : X \to S$.

Important Remarks:

- The construction of

$$f_!$$

  is of subtle global nature,
  as it is related to
  "properness" $=$ "relative compactness" properties.

- The construction of its right adjoint

$$f^!$$

  is of subtle local nature,
  as it is related in classical geometric settings to

$$\begin{cases} \text{relative dimensions,} \\ \text{smoothness or singularities.} \end{cases}$$

- The six operations formalism is powerful enough to
  - generate Lefschetz-type trace "formulas",
  - and so, in the ℓ-adic (or crystalline) case,
    solve the first 3 Weil conjectures.

A general question raised by Grothendieck:

"There is a lack of basic reflection of the following type:
to describe (if possible)
in the context of arbitrary toposes and discrete coefficients (as a first step)
notions of 'properness', 'smoothness', 'local properness', 'separation'
for morphisms of toposes,
allowing to derive a notion
of <u>'admissible morphism'</u> of toposes $f : X \to S$,
for which the two operations $f_!$ and $f^!$ have a meaning (and are adjoint)
so as to obtain the usual properties of the six operations formalism."

A question raised by a formula of Drinfeld:

- Consider

$$\begin{cases} \mathbb{F}_q = \text{finite field with } q \text{ elements}, \\ C = \text{(geometrically connected) smooth projective curve over } \mathbb{F}_q, \\ \mathcal{C} = \text{category of finitely presented schemes over } \mathbb{F}_q, \end{cases}$$

and

$$\begin{array}{ccl} S & \longmapsto & \mathcal{M}(S) = \text{ groupoid of (absolutely irreducible)} \\ \text{object of } \mathcal{C} & & \ell\text{-adic sheaves of rk 2 on } S \times C, \end{array}$$

$$\begin{array}{ccl} S & \longmapsto & \widetilde{\mathcal{M}}(S) = \text{ stackification of } \mathcal{M} \\ & & \text{for the étale topology on } \mathcal{C}. \end{array}$$

Theorem (Drinfeld, using Langlands' correspondence for GL_2):
The series of numbers of points

$$\#\,\widetilde{\mathcal{M}}(\mathbb{F}_{q^n})\,, \quad n \geq 1\,,$$

has the same form as if
$\widetilde{\mathcal{M}}$ was a variety over $\mathbb{F}_q$,
verifying the Weil conjectures !

$\Rightarrow$ **Question**: Does the six operations formalism apply to $\widetilde{\mathcal{M}}$?

The "bridge" between continuous and discrete coefficients provided by some form of non-commutative biduality:

- This is the "Riemann-Hilbert correspondence"
  (= "théorème du bon Dieu")

  established in the late 1970's by Z. Mebkhout, M. Kashiwara.
- Context:

$$\begin{array}{rcl} X & = & \text{complex analytic variety,} \\ \mathcal{O}_X & = & \text{structure sheaf of meromorphic functions on } X, \\ \mathcal{D}_X & = & \text{sheaf of linear differential operators} \\ & = & \text{sheaf of non-commutative rings} \\ & & \text{generated over } \mathcal{O}_X \text{ by derivations,} \\ \mathbb{C}_X & = & \text{center of } \mathcal{D}_X \\ & = & \text{sheaf of locally constant functions.} \end{array}$$

$\longrightarrow$ The correspondence expresses some equivalence, at the level of derived categories, between

- complexes of $\mathcal{D}_X$-Modules (continuous coefficients),
- complexes of $\mathbb{C}_X$-Modules (discrete coefficients).

The equivalence as non-commutative biduality:

Theorem: *The two functors*

$$\left\{ \begin{matrix} \textit{derived category} \\ \textit{of } \mathcal{D}_X\textit{-Modules} \end{matrix} \right\} \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \left\{ \begin{matrix} \textit{derived category} \\ \textit{of } \mathbb{C}_X\textit{-Modules} \end{matrix} \right\}$$

$$\begin{array}{rcl} \mathcal{M} & \longmapsto & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \\ R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{L}, \mathcal{O}_X) & \longleftarrow\!\shortmid & \mathcal{L} \end{array}$$

induce two reverse equivalences

$$\left\{ \begin{matrix} \textit{full subcategory on } \mathcal{D}_X\textit{-complexes} \\ \textit{which are “holonomic”} \\ \textit{with “regular” singularities} \end{matrix} \right\} \begin{matrix} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{matrix} \left\{ \begin{matrix} \textit{full subcategory} \\ \textit{on } \mathbb{C}_X\textit{-complexes} \\ \textit{which are “constructible”} \end{matrix} \right\}.$$

Remark:

There is also a version of that equivalence for

- $X =$ smooth algebraic variety over $\mathbb{C}$,
- $\mathcal{O}_X =$ structure sheaf of algebraic functions on $\mathbb{C}$,
- $\mathcal{D}_X =$ sheaf of algebraic linear differential operators on $\mathbb{C}$.

Grothendieck's dream of "anabelian algebraic geometry":

"I became aware that the daydream I had been
pursuing sporadically for several years,
which had come to be known as 'anabelian algebraic geometry',
was nothing other than a continuation,
an ultimate continuation of Galois theory,
and undoubtedly in the spirit of Galois."

What is this dream ?

- To express part of arithmetic and arithmetic geometry in terms of (ordinary continuous) topology.
- In particular, to present

$$\mathrm{Gal}(\mathbb{Q}) = \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$$

  in purely topological terms.
- To express in purely topological terms rational points of arithmetic varieties or morphisms between arithmetic varieties.

The bridge provided by Grothendieck's fusion of Galois theory and Poincaré fundamental groups:

- Context: Let $X =$ any type of "geometric" object such as

$$\begin{cases} \text{a topological space (in the usual sense),} \\ \text{or a scheme (including } \mathrm{Spec}(k),\ k = \text{field),} \\ \text{or even a topos } \mathcal{E}, \end{cases}$$

which is "locally connected" and "connected".

Let $x =$ a point of X such as

$$\begin{cases} x = & \text{ordinary point of a topological space,} \\ x = & \text{"geometric point" } \mathrm{Spec}(\overline{k}) \to X,\ \overline{k} = \text{alg. closed field,} \\ x = & (\mathcal{E} \xrightarrow{x^*} \mathrm{Set},\ \mathrm{Set} \xrightarrow{x_*} \mathcal{E}) = \text{point of a topos } \mathcal{E}. \end{cases}$$

Let $\mathrm{Cov}_X =$ category of "covers" of X, meaning

$$\begin{cases} \text{locally constant covers in the classical sense,} \\ \text{locally constant étale covers of a scheme } X, \\ \text{locally constant objects of the topos } \mathcal{E}. \end{cases}$$

Let $\pi_1(X, x) =$ group of automorphisms of the fiber functor

$$x^* : \mathrm{Cov}_X \longrightarrow \mathrm{Set}$$

with the induced prodiscrete topology.

- The Grothendieck-Galois equivalence:

Theorem: *The natural functor*

$$Cov_X \longrightarrow \left\{ \begin{array}{c} \textit{category of sets} \\ \textit{endowed with a continuous} \\ \textit{action of } \pi_1(X,x) \end{array} \right\}$$

is an equivalence of categories.

Corollary: *The group of self-equivalences of Cov_X indentifies with*

$$\mathrm{Out}(\pi_1(X,x)) = \mathrm{Aut}(\pi_1(X,x))/\{\textit{inner automorphisms}\}.$$

- The classical cases:

- If $X =$ (locally simply connected and connected) topological space,

$$\begin{array}{rcl} \pi_1(X,x) & = & \text{Poincaré fundamental group} \\ & = & \{\text{homotopy classes of loops at } x\}. \end{array}$$

- If $X = \mathrm{Spec}(k)$, $x = \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$,

$$\pi_1(X,x) = \mathrm{Aut}(\overline{k}/k) = \text{Galois group}.$$

From algebraic geometry to topology:

Let X = algebraic variety over $\mathbb{C}$,
$X(\mathbb{C})$ = associated topological space,
x = element of $X(\mathbb{C})$.

Theorem: *The natural functor*

$$Cov_X \longrightarrow Cov_{X(\mathbb{C})}$$

induces an equivalence

$$\left\{ \begin{array}{c} \textit{finite étale} \\ \textit{covers of } X \\ \textit{as a scheme} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{finite covers} \\ \textit{of } X(\mathbb{C}) \\ \textit{as an ordinary topological space} \end{array} \right\}.$$

Corollary:

$$\begin{array}{ccc} \pi_1(X,x) = & \textit{profinite completion of} & \pi_1(X(\mathbb{C}),x). \\ \| & & \| \\ \textit{algebraic} & & \textit{topological} \\ \textit{fundamental} & & \textit{Poincaré} \\ \textit{group} & & \textit{fundamental group} \end{array}$$

From arithmetic to topology:

- Context: Let $\mathbb{Q} \subset E \subset \overline{\mathbb{Q}} \subset \mathbb{C}$.
$$\underset{\text{number field}}{\overset{E}{\|}}$$
- For X = algebraic variety over E,
  $\overline{X} = X \times_{\mathrm{Spec}(E)} \mathrm{Spec}(\overline{\mathbb{Q}})$ = induced variety over $\overline{\mathbb{Q}}$,
  $X(\mathbb{C})$ = induced ordinary topological space.

  Then
  $$\mathrm{Gal}(E) = \mathrm{Aut}(\overline{\mathbb{Q}}/E)$$
  acts naturally on
  $$\left\{\begin{array}{c}\text{category of}\\ \text{finite étale covers}\\ \text{of } \overline{X}\end{array}\right\} = \left\{\begin{array}{c}\text{category of}\\ \text{finite topological}\\ \text{covers of } X(\mathbb{C})\end{array}\right\}.$$
- Any morphism of algebraic varieties over E
  $$X \longrightarrow S$$
  induces an exact functor
  $$\left\{\begin{array}{c}\text{category of finite topological}\\ \text{covers of } S(\mathbb{C})\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of finite topological}\\ \text{covers of } X(\mathbb{C})\end{array}\right\}$$
  which respects the actions of $\mathrm{Gal}(E)$.

Natural questions:

If $X, S =$ algebraic varieties over E,
consider the natural map

$$\left\{ \begin{array}{c} \text{morphisms} \\ X \to Y \\ \text{defined over } E \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{exact functors} \\ \mathrm{Cov}^f_{S(\mathbb{C})} \to \mathrm{Cov}^f_{X(\mathbb{C})} \\ \text{which respect the action of } \mathrm{Gal}(E) \end{array} \right\}.$$

$\to$ Is it possible to know
in some classes of cases
that this map is

injective ?

surjective ?

one-to-one ?

The Galois group of $\mathbb{Q}$ and the Teichmüller tower:

Starting observations:

- All "multiplicities" $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}_{g,n}}$
  are defined over $\mathbb{Q}$ (and even extend to $\mathrm{Spec}(\mathbb{Z})$),
  as well as all the natural morphisms
  that relate them.
- So $\mathrm{Gal}(\mathbb{Q}) = \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$
  acts on the categories of finite topological covers of

  $$\text{all } \mathcal{M}_{g,n}(\mathbb{C}) \quad \text{and} \quad \overline{\mathcal{M}_{g,n}}(\mathbb{C}),$$

  and these actions respect the natural functors between these categories.
- This action is faithful already for

  $$\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$$

  as a consequence of Belyi's theorem
  (which Grothendieck had conjectured).

Natural questions:

- Consider the embedding

$$\mathrm{Gal}(\mathbb{Q}) \hookrightarrow \left\{ \begin{array}{c} \text{self-equivalences} \\ \text{of the categories of} \\ \text{finite topological covers} \\ \text{of all } \mathcal{M}_{g,n}(\mathbb{C}) \text{ and } \overline{\mathcal{M}_{g,n}}(\mathbb{C}) \\ \text{which respect their natural relations} \end{array} \right\}.$$

$\longrightarrow$ Is this embedding an isomorphism ?

$\longrightarrow$ Does it provide a way
to describe $\mathrm{Gal}(\mathbb{Q})$
in concrete purely topological terms ?