Grothendieck toposes and point-free geometry: a broader intellectual horizon for the representation and the processing of images?

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Note:

This talk is a fruit
of many discussions
between Olivia Caramello and L.L.

Mathematical models and computing processes

- Basic general remark:

  Any computing process or device
  is conceived and realized
  as a concrete and explicit implementation
  of mathematical representations, models or theories.

- Consequences:

  $\longrightarrow$ The more mathematics we know,
  the more computation devices become possible.

  $\longrightarrow$ Our imagination for creating new computing devices
  is limited by the present state of mathematics
  and the present state of our knowledge in math.

Illustrations of this relationship

- The importance of numerical functions in computer use:

  Maybe this is partly because
  most computer scientists and engineers
  have been mainly mathematically educated
  in numerical functions theory?

- The particular case of Deep Learning:

  The mathematical framework for that
  is the theory of approximation of numerical functions
  (on a compact subset of a linear space of large finite dimension)
  as multiple alternate composites of

  - predetermined truncation functions,
  - arbitrary affine functions,
    whose parameter coefficients
    have to be chosen.

The particular but most important case of images

- In modern technology,
  images appear as collection of "pixels"
  indexed by coordinates (which are
  integral multiples of some small length).
- Each "pixel" is a finite family
  of measures of intensity (taking numerical values) of some colors.
- So images appear as vectors
  in some linear spaces
  of large enough finite dimension.

$\longrightarrow$ Consequence:

- Image recognition devices
  consist in approximating functions
  (usually as composites of functions of some given elementary type)
  on some compact subspaces
  of such linear spaces.

The math behind the representation of images by “pixels”

- This type of representation by “pixels”
  is certainly related to the fact that,
  in modern mathematics taught in universities,
  any type of geometric object
  (ex: differential manifold, analytic variety,
  algebraic variety, topological spaces, $\cdots$)
  is always presented as
  - an underlying set
    (called the set of points of the geometric object),
  - plus an extra structure
    (ex: a topology, a metric, a differential, analytic or algebraic structure, $\cdots$)

  which can be introduced
  only once the underlying set of points
  is already specified.

Natural objections to the representation of images by “pixels”

- For our mind, the world does not consist in points.
  In fact, we never see points.
  We see objects
  together with their relative positions
  and the decomposition relations
  of big objects into smaller pieces which are their constituents.
- If images are represented as collections of “pixels”,
  they become vectors in some linear spaces.
  But most vectors in such linear spaces
  do not correspond at all
  to images
  that could actually be seen.

An objection to the objections

- Our mind never sees points
  but
  the cone cells and rod cells in our retina,
  which are our light receptors,
  have similarities with "pixels".

$\longrightarrow$ Consequence:
If we look for a mathematical theory
of geometric objects
which would correspond more closely
to what images are in our mind,
we need a theory where

  - the notion of point is still well-defined,
  - but geometric objects
    are not introduced as sets of points
    possibly endowed with extra structures.

Grothendieck's theory of toposes

- Toposes were introduced by Grothendieck as the most general notion of space which could be defined in mathematics.

→ Indeed, all classical notions of space define toposes but there are many more toposes which do not correspond to any classical notion of geometric object.

Consequences we can hope for:

- Any image should define a topos (or many toposes, corresponding to the level of precision of the chosen representations).
- There should also exist a topos of all images (of some given level of precision).

Points and geometric presentations of toposes

- Any Grothendieck topos
  has a well-defined collection of points,
  but it is not defined by its points.
  (In fact, some very rich toposes do not have any point.)
- Toposes can be geometrically presented
  by some kind of “sketches” (technically called “sites”).
- Any such “sketch” defines a unique topos,
  but any topos can be presented
  by infinitely many “sketches”.
- Toposes are constructed from “sketches”
  by some universal “interpolation” process.
- A sketch represents a topos when it contains
  - a rich enough family of geometric objects
    which are part of the topos,
  - the induced relations between these objects,
  - an induced “glueing device” which allows
    to reconstruct complex objects from smaller pieces.

Linguistic presentations of toposes

- Any Grothendieck topos
  also admits (infinitely many) linguistic descriptions
  consisting in

  $\begin{cases} - \quad \text{names of geometric objects,} \\ - \quad \text{names of relations between objects,} \end{cases}$

  supplemented by

  - grammar rules (called “axioms”).
- In the reverse direction,
  any such language (= names + grammar rules)
  defines a topos
  which geometrically incarnates
  the “semantics” of this language
  i.e. everything it can express.

$\rightarrow$ Consequence:

- Any image should have (infinitely many) linguistic descriptions.
- If there exists a topos of all images,
  there should also exist a theory (or infinitely many) theories of images.

The basic diagram of topos theory