The metamorphosis of the notion of space, according to Grothendieck

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The word “metamorphosis” and the theme of transformation

- The word “metamorphosis’” appears in “Récoltes et Semailles”
  - in a mathematical sense,
  - in a human life sense.
- In a mathematical sense:
  “the metamorphosis of the notion of space”.
- In a human life sense:
  the metamorphosis of a human person along the years.

Remark. – A metamorphosis can be negative:
the loss of child-like innocence,
in particular when a person becomes well-recognized and powerful.

- The importance of the theme of transformation, in “La clé des Songes”:
  - → Human vocation consists in
    transforming one-self every day,
    in the sense of deeper and deeper humanization.
  - → The difference between “information” and “knowledge” is that
    – information does not change us internally,
    – knowledge transforms us.

The double metamorphosis of the notion of space

- First metamorphosis:
  From algebraic varieties to schemes.
- Second deeper metamorphosis:
  From topological spaces to toposes.
- The two metamorphoses follow the same lines:
  - Both are made possible by the notion of sheaf, due to Leray (as modified by Cartan).
  - Both are based on an embedding of a classical concrete geometric world into a more abstract world.

{ world of algebraic varieties } —Serre / Zariski topology + structure sheaf→ { world of spaces endowed with a sheaf of rings }

{ world of topological spaces } —Grothendieck / category of all set-valued sheaves→ { apparently "discrete" world of categories }

Serre's structure sheaf of an algebraic variety

- An affine algebraic variety over an algebraically closed field K is a subset of some K^n which is defined by polynomial equations P_i in n variables

$$V = \{(k_1, \cdots, k_n) \in K^n \mid P_i(k_1, \cdots, k_n) = 0\,, \quad 1 \leq i \leq k\}.$$

- The Zariski topology on V is the smallest topology such that, for any polynomials $f_1, \cdots, f_m$,

$$V_{f_1,\cdots,f_m} = \{(k_1, \cdots, k_n) \in V \mid f_j(k_1, \cdots, k_m) \neq 0\,, \quad 1 \leq j \leq m\}$$

is an open subset.

- Serre's structure sheaf: the unique sheaf which sends

$$\begin{aligned}
\mathcal{O}_V \quad : \qquad V \quad &\longmapsto \quad A_V = K[X_1, \cdots, X_n] / \left\{ \begin{array}{c} \text{ideal of polynomials} \\ \text{which vanish on } V \end{array} \right\}, \\
V_{f_1,\cdots,f_m} \quad &\longmapsto \quad A_{V_{f_1,\cdots,f_m}} \\
&= \quad \text{localization of } A_V \text{ along } f_1, \cdots, f_m \\
&= \quad A_V[Y_1, \cdots, Y_m] / \left\{ \begin{array}{l} \text{ideal generated by} \\ \text{polynomials } Y_j \cdot f_j - 1, 1 \leq j \leq m \end{array} \right\}.
\end{aligned}$$

Serre's embedding

$$\left\{\begin{array}{c}\text{world of}\\ \underline{\text{algebraic varieties}}\\ \text{over an } \underline{\text{alg. closed field K}}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{world of}\\ \underline{\text{topological spaces}}\\ \text{endowed with}\\ \text{a } \underline{\text{sheaf of rings}}\end{array}\right\}$$

$$V \longmapsto \left\{\begin{array}{c}\text{set of } \underline{\text{points}} \text{ of } V\\ + \underline{\text{Zariski topology}}\\ + \underline{\text{structure sheaf}} \text{ of } V\end{array}\right\}$$

$$\begin{array}{c}(V \xrightarrow{p} W)\\ \|\\ \text{algebraic map}\\ \text{defined by}\\ \underline{\text{polynomials}}\end{array} \longmapsto \left\{\begin{array}{c}\text{Zariski } \underline{\text{continuous map}}\\ p : V \to W\\ \text{completed with a } \underline{\text{morphism of sheaves of rings}}\\ \mathcal{O}_W \to p_*\mathcal{O}_V\end{array}\right\}$$

Remark. – According to Hilbert's Nullstellensatz,

$$\left\{\begin{array}{c}\underline{\text{points}}\\ \text{of } V\end{array}\right\} = \left\{\begin{array}{c}\underline{\text{maximal ideals}}\\ \text{of } A_V = \mathcal{O}_V(V)\end{array}\right\}.$$

From algebraic varieties to schemes

Grothendieck realized that
Serre's construction could be adapted
to define an embedding

$$\left\{\begin{array}{c}\text{world of}\\ \text{commutative rings}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{world of}\\ \text{topological spaces } X\\ + \text{ sheaf of rings } \mathcal{O}_X\end{array}\right\}$$

$$\begin{array}{rcl} A & \longmapsto & \mathrm{Spec}(A) \\ & = & \text{set of points} \\ & & + \text{ (Zariski) topology} \\ & & + \text{ structure sheaf of rings } \mathcal{O}_A \end{array}$$

$$\begin{array}{c}(A \xrightarrow{u} B)\\ \| \\ \text{homomorphism}\\ \text{of}\\ \text{commutative rings}\end{array} \longmapsto \left\{\begin{array}{c}\text{continuous map}\\ \mathrm{Spec}(B) \xrightarrow{p} \mathrm{Spec}(A)\\ \text{supplemented with a morphism}\\ \text{of sheaves of rings}\\ \mathcal{O}_A \to p_*\mathcal{O}_B\end{array}\right\}$$

The metamorphosis of points

- By definition, for any $A =$ commutative ring,
  the points of $\mathrm{Spec}(A)$
  are the prime ideals of A
  (not the maximal ideals).
- This is why any homomorphism

$$A \xrightarrow{u} B$$

defines a (continuous) map

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(B)\,, \\ p & \longmapsto & u^{-1}(p) \\ \| & & \| \\ \text{prime ideal of } A & & \text{prime ideal of } B. \end{array}$$

Remarks. –

- $\mathrm{Spec}(\mathbb{Z})$ is well-defined as a "scheme" whose points are
  (0) and prime integers.
- Arithmetic and algebraic geometry
  are unified in scheme theory.

Sheaves of modules

- If X is a topological space endowed with a sheaf of rings $\mathcal{O}_X$, there is an induced notion of $\mathcal{O}_X$-Module ($=$ sheaf of modules) on X.
- They are "presheaves" of modules $\mathcal{M}$

$$\begin{cases} \begin{array}{ccc} U & \longmapsto & \mathcal{M}(U) \\ \| & & \| \\ \text{open subset of } X & & \text{module over the ring } \mathcal{O}_X(U), \\ & & \\ (V \subset U) & \longmapsto & (\mathcal{M}(U) \to \mathcal{M}(V)) \\ & & = \mathcal{O}_X(U)\text{-linear restriction map}, \end{array} \end{cases}$$

  which verify the "glueing" sheaf conditions.
- In the case of algebraic varieties $(V, \mathcal{O}_V)$, Serre considered some particular families of $\mathcal{O}_V$-Modules,

  "coherent and quasi-coherent $\mathcal{O}_V$-Modules"

  and studied their "cohomology".

Sheaf cohomology

- Serre constructed cohomology of coherent or quasi-coherent $\mathcal{O}_V$-Modules on algebraic varieties $(V, \mathcal{O}_V)$ as "Čech cohomology".
- Grothendieck's Tohoku paper:
  Construction of cohomology for arbitrary left exact linear functors

$$F : \mathcal{M}od_{\mathcal{O}_X} = \left\{ \begin{array}{c} \text{category of} \\ \mathcal{O}_X\text{-Modules} \\ \text{on } X \end{array} \right\} \longrightarrow \mathcal{A}$$

  on categories of sheaves of modules on

  X = topological space, endowed with $\mathcal{O}_X$ = sheaf of rings on X.
- He identifies the categorical properties of all categories of sheaves of modules $\mathcal{M}od_{\mathcal{O}_X}$ which make cohomology well-defined:
  - $\longrightarrow$ They are abelian varieties.
  - $\longrightarrow$ They verify a short list of extra properties:
    - – arbitrary sums are well-defined,
    - – filtering colimit functors are exact,
    - – the category admits a generator.

Two families of examples

- First family:
  The categories of $\mathcal{O}_X$-Modules
  $$\mathcal{M}od_{\mathcal{O}_X}$$
  for any sheaf $\mathcal{O}_X$ of rings
  on any topological space X.
- Second family: "categories of diagrams"
  later called "categories of presheaves".
  They are associated to
  $\mathcal{C}$ = small category,
  $(\mathcal{O} : \mathcal{C}^{\mathrm{op}} \to \mathcal{R}ing)$ = contravariant functor
  $$\begin{cases} \text{object } X & \longmapsto \quad \mathcal{O}(X) = \text{ring}, \\ \text{morphism } (X \xrightarrow{u} Y) & \longmapsto \quad (\mathcal{O}(u) : \mathcal{O}(Y) \to \mathcal{O}(X)) = \text{ring homomorphism.} \end{cases}$$
  They consist in the categories
  $\mathcal{M}od_{\mathcal{O}}$ of contravariant functors $\mathcal{M}$
  $$\begin{cases} \text{object } X & \longmapsto \quad \mathcal{M}(X) = \mathcal{O}(X)\text{-module}, \\ (X \xrightarrow{u} Y) & \longmapsto \quad (\mathcal{M}(u) : \mathcal{M}(Y) \to \mathcal{M}(X)) = \mathcal{O}(Y)\text{-linear map.} \end{cases}$$

Particular case: Categories of linear representations
of a group or a monoid.

Unifying the two families of examples into a single family

- Grothendieck remarked
  that the notion of sheaf (of rings, of modules, $\cdots$)
  can be defined in the general setting of
  a "site" $(\mathcal{C}, J)$ consisting in
  $$\begin{cases} \mathcal{C} = \text{(essentially) small category,} \\ J = \text{"topology" on } \mathcal{C} = \text{coherent notion of "covering".} \end{cases}$$
- The first family of examples is the particular case when
  $$\begin{cases} \mathcal{C} = \text{category of open subspaces of a topological space } X, \\ J = \text{canonical topology.} \end{cases}$$
- The second family of examples is the particular case when
  $$J = \text{discrete topology on } \mathcal{C} = \text{small category.}$$
- In general, if
  $(\mathcal{C}, J) = \text{site}$
  $\mathcal{O} = \text{sheaf of rings on } (\mathcal{C}, J)$,
  the associated category of sheaves of $\mathcal{O}$-modules
  $$\mathcal{M}od_{\mathcal{O}}$$
  is a Grothendieck abelian category,
  so that cohomology is well-defined there.

Set-valued sheaves and toposes

- Grothendieck decided to consider not only
  sheaves of rings, of modules, $\cdots$
  but, most basically sheaves of sets.
- The notion of "sheaf of sets" or "set-valued sheaf"
  makes sense on any site $(\mathcal{C}, J)$.
  It is a presheaf
  $$P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$
  which verifies some "glueing conditions" with respect to J-coverings on $\mathcal{C}$.
- The sheaves of sets on a site $(\mathcal{C}, J)$ make up
  a (locally small) category $\widehat{\mathcal{C}}_J$.

Definition. – *A topos is a category which is equivalent to the category of sheaves*

$$\widehat{C}_J \quad \textit{on some site } (\mathcal{C}, J)\,.$$

Remark. – Completely different sites $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ may define equivalent toposes $\widehat{\mathcal{C}}_J \cong \widehat{\mathcal{C}'}_{J'}$.

The metamorphosis of topological spaces

- Associate to any topological space X the topos $\mathcal{E}_X$ of set-valued sheaves on X.

Proposition. –

(i) *Any continuous map between topological spaces*

$$Z \xrightarrow{z} X$$

induces a pair of adjoint functors

$$(\mathcal{E}_X \xrightarrow{z^*} \mathcal{E}_Z \, , \, \mathcal{E}_Z \xrightarrow{z_*} \mathcal{E}_X)$$

such that z^* respects finite limits.

(ii) *Conversely, if X is "sober", any pair of adjoint functors,*

$$(\mathcal{E}_X \xrightarrow{z^*} \mathcal{E}_Z \, , \, \mathcal{E}_Z \xrightarrow{z_*} \mathcal{E}_X)$$

such that z^* respects finite limits,
is induced by a unique continuous map $\quad z : Z \longrightarrow X$.

Remarks. –

- This applies in particular if $Z = \{\bullet\}$ and $\mathcal{E}_Z = \mathrm{Set}$.
- Most topological spaces of concrete use are "sober".
  Any topological space X has a canonical "soberification"

$$X \longrightarrow |X| \qquad \text{inducing} \qquad \mathcal{E}_X \xrightarrow{\sim} \mathcal{E}_{|X|} \, .$$

The metamorphosis of topology

Definition. –

(i) *A morphism of toposes* $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ *is a pair of adjoint functors*

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}' \, , \, \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f^* *respects finite limits.*

(ii) *A point of a topos* $\mathcal{E}$ *is a morphism of toposes* $\quad \mathrm{Set} \longrightarrow \mathcal{E}$.

(iii) *A subtopos of a topos* $\mathcal{E}$ *is a morphism of toposes*

$$(\mathcal{E}' \xrightarrow{f} \mathcal{E}) = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}' \, , \, \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f_* *is "fully faithful".*

Remarks. –

- Any subspace Z of a topological space X defines a subtopos

$$\mathcal{E}_Z \hookrightarrow \mathcal{E}_X$$

  but, in general, $\mathcal{E}_X$ has subtoposes which do not come from subspaces.
- Subtoposes of a given topos $\mathcal{E}$ make up a partially ordered set, with arbitrary unions and intersections.
- It is also possible to define open and closed subtoposes. In the case of a topos $\mathcal{E}_X$, they correspond to open and closed subsets of X.

Toposes as an “unsuspected generalisation” of the notion of space

- As an illustration, consider a measure on a topological space X

$$\mu : \{\text{open subsets of } X\} \longrightarrow \mathbb{R}^+$$

which verifies

$$\mu(U \cup V) = \mu(U) + \mu(V) - \mu(U \cap V) \quad \text{for any} \quad U, V$$

$$\mu\left(\bigcup_{i \in I} U_i\right) = \sup_i \mu(U_i) \quad \text{for any filtering family} \quad (U_i)_{i \in I}\,.$$

Banach-Tarski paradox. –
In general, μ cannot be extended to a measure of all subsets

$$\{\textit{subsets of } X\} \longrightarrow \mathbb{R}^+$$

verifying the same properties as μ.

Theorem (Olivier Leroy). –
In general, μ extends naturally to a map

$$\{\textit{subtoposes}\} \longrightarrow \mathbb{R}^+$$

verifying the same properties as μ.

Explanation of the paradox:
Some subsets $Y \hookrightarrow X$ and $Z \hookrightarrow X$ may verify $\quad Y \cap Z = \emptyset$
but $\mathcal{E}_Y \cap \mathcal{E}_Z =$ non trivial topos with non zero measure.

Toposes as "pastiches" of the category of sets

- Toposes have been defined in a constructive way as

  categories equivalent to some $\widehat{\mathcal{C}}_J = \{$sheaves on a site $(\mathcal{C}, J)\}$.
- The simplest non trivial topos is

  Set $=$ topos of sheaves on the point space $\{\bullet\}$.
- Toposes can also be charaterized in an axiomatic way:

Giraud's Theorem. – *Toposes are categories which have the same constructive categorical properties as Set:*

- *They are locally small.*
- *They have arbitrary limits and colimits.*
- *Fiber product functors respect colimits.*
- *Filtering colimit functors respect finite limits.*
- *Quotients correspond to equivalence relations.*
- *Subobjects and quotient objects of any object form a set.*
- *Sums are disjoint.*
- *Exponentials $B^A = \mathcal{H}om(A, B)$ are well-defined.*

Consequence. – Any constructive math theory (which doesn't make use of the axiom of choice or of the law of excluded middle) can be developed in the context of an arbitrary topos as well as in the context of sets.

A notion which is "wide enough but not too wide"

- The notion of topos is wide enough so that
  - any topological space defines a topos,
  - any classical geometric object defines a topos (or several toposes) usually endowed with an extra structure such as an inner ring (= sheaf of rings),
  - it allows to define and study with geometric intuitions completely new objects of topological type such as the étale and crystalline toposes in algebraic geometry.
- The notion of topos is very stable under natural constructions such as
  - "localization": for any object E of a topos $\mathcal{E}$, the relative category $\mathcal{E}/E = \{\text{category of morphisms } E' \to E\}$ is a topos,
  - "classifying actions of inner groups": for any inner group G of a topos $\mathcal{E}$, the category of its actions $BG = \{\text{category of objects } E \text{ of } \mathcal{E} \text{ endowed with an action } G \times E \to E\}$ is a topos.
- The notion of topos is "not too wide" as
  - the usual vocabulary of topology and geometry still makes sense in the context of any topos,
  - geometric intuitions still apply in the general context of toposes,
  - usual topological invariants (in particular cohomology) are defined for arbitrary toposes.

The "double bed" of the continuous and the discrete

- The notion of topos is "wide enough"
  to take care of "continuous structures" as,
  for any topological space X,

$$\mathcal{E}_X = \{\text{topos of sheaves on } X\} \quad \text{is a topos.}$$

- It is also "wide enough"
  to take care of "discrete structures" as,
  for any small category $\mathcal{C}$,

$$\widehat{\mathcal{C}} = \{\text{category of presheaves on } \mathcal{C}\} \quad \text{is a topos.}$$

- As it unifies continuous and discrete structures
  in a unique framework,
  it allows to introduce and study
  "intermediate structures"
  such as the
  étale and crystalline topologies
  of arithmetic algebraic geometry.

Points of toposes

- Points of a topos $\mathcal{E}$ are toposes morphisms $\quad \mathrm{Set} \longrightarrow \mathcal{E}$.
  They make up a category $\mathrm{pt}(\mathcal{E}) = [\mathrm{Set}, \mathcal{E}]_\top$.
- More generally, for any topos $\mathcal{E}'$, the category
  $$[\mathcal{E}', \mathcal{E}]_\top = \{\text{category of toposes morphisms } \mathcal{E}' \to \mathcal{E}\}$$
  can be called the category of $\mathcal{E}'$-valued points of $\mathcal{E}$.
- Any toposes morphism $\mathcal{E}'' \xrightarrow{e} \mathcal{E}'$ defines a composition functor
  $$[\mathcal{E}', \mathcal{E}]_\top \xrightarrow{\bullet \circ e} [\mathcal{E}'', \mathcal{E}]_\top \,.$$

Theorem (consequence of Diaconescu's equivalence). –
Any choice of a presenting site $(\mathcal{C}, J) \qquad \mathcal{E} \cong \widehat{\mathcal{C}}_J$
defines a first-order (geometric) theory $\mathbb{T}$
with natural interpretations for arbitrary toposes $\mathcal{E}'$
$$[\mathcal{E}', \mathcal{E}]_\top \xrightarrow{\sim} \mathbb{T}\text{-}mod(\mathcal{E}') = \{\textit{category of models of the theory } \mathbb{T} \textit{ in } \mathcal{E}'\}.$$

Remarks. –

- In topos theory, the notion of point is a derived notion,
  not a defining notion as in classical geometry.
- The consideration of points of toposes
  brings from geometry to linguistic descriptions (= logic).

Classifying toposes

- Conversely, consider a first-order theory $\mathbb{T}$ which is "geometric": its axioms only involve colimits and finite limits.
- Then, $\mathbb{T}$ defines for any topos $\mathcal{E}$ a category

$$\mathbb{T}\text{-mod}(\mathcal{E}) = \{\text{category of models of } \mathbb{T} \text{ in } \mathcal{E}\}$$

and for any toposes morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ a functor

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

Theorem. – *For any such first-order geometric theory $\mathbb{T}$, there exists a topos $\mathcal{E}_{\mathbb{T}}$ (unique up to canonical equivalence) endowed with a "universal model" $M_{\mathbb{T}}$ of $\mathbb{T}$ such that, for any topos $\mathcal{E}$, the functor*

$$\begin{array}{ccc} [\mathcal{E}, \mathcal{E}_{\mathbb{T}}]_{\top} & \longrightarrow & \mathbb{T}\text{-}mod(\mathcal{E})\,, \\ (\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* M_{\mathbb{T}} \end{array}$$

is an equivalence of categories.

Remarks. –

- The "classifying topos" $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ can be constructed explicitly as

  $\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})_{J_{\mathbb{T}}}}$ where $\mathcal{C}_{\mathbb{T}} =$ category of "formulas" of $\mathbb{T}$.
- The construction $\mathbb{T} \mapsto \mathcal{E}_{\mathbb{T}}$ goes from language to geometry. It provides a canonical geometric incarnation of the semantics of any theory.

Relative toposes and "moduli" problems

- Consider a site $(\mathcal{C}, J)$.
- Consider a "moduli" problem consisting in associating
  - to any object X of $\mathcal{C}$, some $P(X)$ which is the set of isomorphism classes (or the category) of geometric structures of some type over X (ex: any kind of manifolds, varieties or schemes fibered over X, vector bundles of some rank over X, principal bundles of some group G over X),
  - to any morphism $X' \xrightarrow{x} X$ of $\mathcal{C}$, a map or a functor
  $$x^* : P(X) \longrightarrow P(X')$$
  defined by some kind of fiber product over $X' \xrightarrow{x} X$
  $$X' \times_X \bullet \, .$$
- If P is set-valued, it is a presheaf and may be a sheaf. If P is category-valued, it may be a "stack".

Proposition. – *Whatever P, it defines a relative topos* $\mathcal{E}_P \longrightarrow \widehat{\mathcal{C}}_J$ *which can be called the "classifying topos" of the kind of geometric structures under consideration.*