III. Presheaf type theories

Reminder of the definition and basic examples:

Definition. – *A first-order geometric theory* $\mathbb{T}$ *is said to be "presheaf type" if its classifying topos*

$$\mathcal{E}_{\mathbb{T}}$$

is equivalent to a topos of presheaves $\widehat{\mathcal{C}}$ *on an (essentially) small category* $\mathcal{C}$.

Examples of presheaf type theories:

- the "empty" theory (i.e. without axioms) on any signature Σ,
- algebraic theories,
- more generally "Horn" theories,
- more generally still Cartesian theories,
- the "theory of flat functors"

  $$\mathbb{T}_{\mathcal{C}}^{p}$$

  on any small category $\mathcal{C}$ (whose models in any topos $\mathcal{E}$ are flat functors

  $$\mathcal{C} \longrightarrow \mathcal{E}\ ).$$

Presheaf-type theories as bases for construction of first-order geometric theories:

We deduce from the given examples:

Corollary. –
Let $\mathbb{T}$ be a first-order geometric theory of signature Σ.
Let $\mathbb{T}_0$ be any Cartesian theory with the same signature Σ whose axioms are provable in $\mathbb{T}$.
Then $\mathbb{T}$ appears as a quotient theory of the presheaf type theory $\mathbb{T}_0$.

Note. –
Consequently, the classifying topos of $\mathbb{T}$ is written as the topos of sheaves

$$(\widehat{\mathcal{C}^{\mathrm{car}}_{\mathbb{T}_0}})_{J_{\mathbb{T}}} \cong \mathcal{E}_{\mathbb{T}}$$

on the Cartesian syntactic category of $\mathbb{T}_0$

$$\mathcal{C}^{\mathrm{car}}_{\mathbb{T}_0}$$

endowed with a certain Grothendieck topology

$$J_{\mathbb{T}}$$

defined by the axioms of $\mathbb{T}$ which are not provable in $\mathbb{T}_0$.

Geometric presentations of classifying toposes and associated presheaf type theories:

We deduce from the "duality theorem" between Grothendieck topologies and quotient theories:

Corollary. –
Let $\mathbb{T}$ be a first-order geometric theory.
Consider a presentation of its classifying topos

$$\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}}$$

as the topos of sheaves on a small category $\mathcal{C}$ equipped with a topology J.
Let $\mathbb{T}_0$ be any geometric theory such that

$$\widehat{\mathcal{C}} \cong \mathcal{E}_{\mathbb{T}_0} .$$

Then $\mathbb{T}$ appears as semantically equivalent (or Morita-equivalent) to a quotient theory $\mathbb{T}'$ of $\mathbb{T}_0$ such that

$$\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}'} .$$

Remark. – In particular, we can take for $\mathbb{T}_0$ the theory

$$\mathbb{T}_{\mathcal{C}}^{p}$$

of "flat functors" on $\mathcal{C}$.

Models of presheaf type theories:

In order to understand the specificity of presheaf type theories,
we start by looking at their set-based models:

Proposition. –
Let $\mathbb{T}$ be a presheaf type theory.
Then any equivalence of toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

(for an essentially small category $\mathcal{C}$)
induces an equivalence of categories

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set})$$

from the "category of ind-objects" of the category $\mathcal{C}^{\mathrm{op}}$ opposite to $\mathcal{C}$
to the category of set-based models of $\mathbb{T}$.

Note. – In particular, any equivalence

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

induces a fully faithful functor

$$\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathbb{T}\text{-mod}(\mathrm{Set}) \,.$$

The notion of category of ind-objects:

We recall:

Definition. – *Let $\mathcal{D}$ be an essentially small category. We denote*

$$\mathrm{Ind}(\mathcal{D})$$

the full subcategory of

$$\widehat{\mathcal{D}} = [\mathcal{D}^{\mathrm{op}}, \mathrm{Set}]$$

consisting in functors

$$P : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

which are "ind-objects"
in the sense that they verify the following three equivalent properties:

(1) *P is written as a filtering colimit of representable objects of $\widehat{\mathcal{D}}$.*

(2) *The "category of elements" of P*

$$\int P = \mathcal{D}/P$$

is filtering.

(3) *The functor*

$$P : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

is flat, which means that its extension by colimits

$$\widehat{P} : \widehat{\mathcal{D}^{\mathrm{op}}} \longrightarrow \mathrm{Set}$$

respects finite limits.

The equivalence of the 3 conditions to be an ind-object:

We recall that the "category of elements" of P

$$\int P = \mathcal{D}/P$$

is the category of pairs (X, x) consisting of

$$\begin{cases} \bullet \quad \text{an object } X \text{ of } \mathcal{D}, \\ \bullet \quad \text{an element } x \in P(X) \text{ seen as a morphism of } \widehat{\mathcal{D}} \\ \qquad\qquad y(X) \longrightarrow P\,. \end{cases}$$

$(2) \Rightarrow (1)$ because we have in $\widehat{\mathcal{D}}$ the formula

$$P = \varinjlim_{(X,x)\in\int P} y(X)\,.$$

$(1) \Rightarrow (3)$ because

$$\begin{cases} \bullet \quad \text{for any object } X \text{ of } \mathcal{D}, \text{ the evaluation functor at } X \\ \qquad\qquad \widehat{\mathcal{D}^{\mathrm{op}}} \longrightarrow \mathrm{Set} \\ \quad\ \text{respects all colimits and all limits}, \\ \bullet \quad \text{in } \mathrm{Set}, \text{ the filtering colimit functors} \\ \qquad\qquad \text{respect finite limits}. \end{cases}$$

(3) $\Rightarrow$ (2) because

- for all objects of $\int P$

$$(X,x) \quad \text{and} \quad (Y,y)\,,$$

  the formula

  $\widehat{P}(y(X) \times y(Y)) = \widehat{P}(y(X)) \times \widehat{P}(y(Y)) = P(X) \times P(Y)$

  shows that there exist an object

$$(Z,z) \quad \text{of} \quad \int P$$

  and two morphisms of $\mathcal{D}$

$$X \longrightarrow Z \longleftarrow Y$$

  which send

$$z \longmapsto x \quad \text{and} \quad z \longmapsto y\,,$$

- for any pair of morphisms of $\int P$

$$(X,x) \overset{u}{\underset{v}{\rightrightarrows}} (Y,y)\,,$$

  the formula

$$\widehat{P}(\ker(y(Y) \rightrightarrows y(X))) = \ker(P(Y) \rightrightarrows P(X))$$

  shows that there exists a morphism of $\int P$

$$(Y,y) \xrightarrow{w} (Z,z)$$

  such that

$$w \circ u = w \circ v\,.$$

Computation of set-based models by a topos-theoretic bridge:

- We compute the topos invariant

$$\mathcal{E} \longmapsto \mathrm{pt}(\mathcal{E}) = [\mathrm{Set}, \mathcal{E}]_{\top}$$

on the two sides of the equivalence of toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \, .$$

- On the side of the classifying topos $\mathcal{E}_{\mathbb{T}}$,
we have a canonical equivalence of categories

$$\mathrm{pt}(\mathcal{E}_{\mathbb{T}}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set}) \, .$$

- On the side of the presheaf topos $\widehat{\mathcal{C}}$,
we are reduced to showing
that there is a canonical equivalence

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{pt}(\widehat{\mathcal{C}}) \, .$$

The category of points of a topos of presheaves:

Proposition. – *Let $\mathcal{C}$ be an essentially small category.*

(i) *For any object X of $\mathcal{C}$, the evaluation at X of presheaves on $\mathcal{C}$*

$$P \longmapsto P(X) \qquad \text{and its right adjoint}$$

$$\begin{array}{rcl} \mathrm{Set} & \longrightarrow & \widehat{\mathcal{C}}\,, \\ I & \longmapsto & P_I = [Y \mapsto \mathrm{Hom}(\mathrm{Hom}(X,Y),I)] \end{array}$$

define a point of the topos $\widehat{\mathcal{C}}$.

(ii) *Associating to any object X of $\mathcal{C}$ the corresponding point of $\widehat{\mathcal{C}}$ defines a fully faithful functor*

$$\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{pt}(\widehat{\mathcal{C}})\,.$$

(iii) *This functor extends to a canonical equivalence*

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{pt}(\widehat{\mathcal{C}})\,.$$

Proof. –

(i) The evaluation functor $P \mapsto P(X)$ respects limits and colimits.

(ii) results from Yoneda's lemma.

(iii) According to Diaconescu's equivalence,

the category $\mathrm{pt}(\widehat{\mathcal{C}})$ is equivalent to that of flat functors $\mathcal{C} \longrightarrow \mathrm{Set}$.

The “finitely presentable” models of presheaf type theories:

Let’s announce the following result:

Theorem. – *Let $\mathbb{T}$ be a presheaf type theory.*
Then any equivalence of toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

induces an equivalence of categories

$$\mathrm{Kar}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set})_{\mathrm{fp}}$$

between

- *the “Karoubi completion”*

$$\mathrm{Kar}(\mathcal{C}^{\mathrm{op}}) \quad \textit{of} \quad \mathcal{C}^{\mathrm{op}},$$

- *the full subcategory*

$$\mathbb{T}\text{-mod}(\mathrm{Set})_{\mathrm{fp}} \hookrightarrow \mathbb{T}\text{-mod}(\mathrm{Set})$$

  made up of the set-based models of $\mathbb{T}$
  which are “finitely presentable”.

Remark. – A set-based model of $\mathbb{T}$ is said to be “finitely presentable” if it is a “compact object” of the category $\mathbb{T}\text{-mod}(\mathrm{Set})$.

The notion of Karoubi completion of a category:

Definition. – *Let $\mathcal{D}$ be a locally small category. We call "Karoubi completion" of $\mathcal{D}$ the category*

$$\mathrm{Kar}(\mathcal{D})$$

of which

- *the objects are the pairs (X, p) formed*
  - *of an object X of $\mathcal{D}$*
  - *of an idempotent $p : X \to X$, with $p \circ p = p$,*
- *the morphisms $(X, p) \to (Y, q)$ are the morphisms of $\mathcal{D}$*

$$u : X \longrightarrow Y \qquad \text{such that} \qquad q \circ u = u \circ p = u.$$

Remarks. –

(i) We always have $\mathrm{Kar}(\mathcal{D})^{\mathrm{op}} = \mathrm{Kar}(\mathcal{D}^{\mathrm{op}})$.

(ii) We have a fully faithful canonical functor $\mathcal{D} \hookrightarrow \mathrm{Kar}(\mathcal{D})$.

(iii) If this functor is an equivalence, we say that $\mathcal{D}$ is "Karoubi-complete".

(iv) The category $\mathrm{Kar}(\mathcal{D})$ is always Karoubi-complete.

(v) If $\mathcal{D}$ is essentially small, the functor

$$\begin{array}{ccc} \mathrm{Kar}(\mathcal{D}) & \longrightarrow & \widehat{\mathcal{D}}, \\ (X, p) & \longmapsto & \ker\big(y(X) \overset{p}{\underset{\mathrm{id}}{\rightrightarrows}} y(X)\big) \end{array}$$

is fully faithful.

The notion of compact object of a category with filtering colimits:

Definition. –
Let $\mathcal{M}$ be a locally small category

which has "arbitrary filtering colimits"

in the sense that for any small filtering category $\mathcal{I}$,

the composition functor with $\mathcal{I} \to \{\bullet\}$

$$\mathcal{M} \longrightarrow [\mathcal{I}, \mathcal{M}]$$

admits a left adjoint

$$\varinjlim_{\mathcal{I}} : [\mathcal{I}, \mathcal{M}] \longrightarrow \mathcal{M} \, .$$

Then an object M of $\mathcal{M}$ is said

"compact"

if the functor

$$\mathrm{Hom}(M, \bullet) : \mathcal{M} \longrightarrow \mathrm{Set}$$

respects all functors of filtering colimits

$$\varinjlim_{\mathcal{I}} \, .$$

Filtering colimits in model categories:

Lemma. – *Let $\mathbb{T}$ be a first-order geometric theory.*
Let $\mathcal{E}$ be a topos.
Let $\mathcal{I}$ be a small filtering category.
Then the functor of filtering colimit

$$\varinjlim_{\mathcal{I}}$$

is well defined in the model category

$$\mathbb{T}\text{-mod}(\mathcal{E})\,.$$

Proof. – Indeed, the filtering colimit functor

$$\varinjlim_{\mathcal{I}}$$

is well defined in the topos $\mathcal{E}$, and it respects

- arbitrary finite limits,
- arbitrary colimits,
- so also the interpretations
  of geometric formulas of the signature Σ of $\mathbb{T}$.

The notion of compact object in model categories:

Corollary. – *Let $\mathbb{T}$ be a first-order geometric theory.*
Let $\mathcal{E}$ be a topos.
Then:

(i) *The functors of filtering colimits*

$$\varinjlim_{\mathcal{I}}$$

are well defined in the category $\mathbb{T}\text{-mod}(\mathcal{E})$.

(ii) *The notion of compact object M is well defined by requiring that the functor*

$$\mathrm{Hom}(M, \bullet) : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathrm{Set}$$

respects all filtering colimits.

Definition. – *A set-based model M of a first-order geometric theory $\mathbb{T}$ is said to be "finitely presentable" if it is a compact object of the category*

$$\mathbb{T}\text{-mod}(\mathrm{Set}) \, .$$

Computation of finitely presentable set-based models by a bridge:

- We start from an equivalence of toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

and the equivalence of categories that it induces

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set})\,.$$

- Considering this last equivalence, we calculate on both sides the full subcategories of compact objects.
- On the side of $\mathbb{T}\text{-mod}(\mathrm{Set})$, we find by definition the full subcategory of finitely presentable models

$$\mathbb{T}\text{-mod}(\mathrm{Set})_{\mathrm{fp}}\,.$$

- It remains to determine the compact objects of the category with filtering colimits

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})\,.$$

Determination of compact ind-objects:

Lemma. – *Let $\mathcal{D}$ be an essentially small category.*
Then the fully faithful functor

$$\begin{array}{ccc} \mathrm{Kar}(\mathcal{D}) & \hookrightarrow & \widehat{\mathcal{D}}\,, \\ (X,p) & \longrightarrow & \ker(y(X) \overset{p}{\underset{\mathrm{id}}{\rightrightarrows}} y(X)) \end{array}$$

is an equivalence onto the full subcategory of

$$\mathrm{Ind}(\mathcal{D})$$

consisting of compact objects.

Proof. –

- For any object X of $\mathcal{D}$ equipped with an idempotent
$$p : X \longrightarrow X \quad \text{verifying} \quad p \circ p = p\,,$$
the subcategory of $\widehat{\mathcal{D}}$ made up of
the object $y(X)$ equipped with the two morphisms p, id
is filtering, and its colimit is the image of
$$(X,p) \quad \text{by} \quad \mathrm{Kar}(\mathcal{D}) \hookrightarrow \widehat{\mathcal{D}}\,.$$
So we have a factorization $\quad \mathrm{Kar}(\mathcal{D}) \hookrightarrow \mathrm{Ind}(\mathcal{D})\,.$

- Objects X of $\mathcal{D}$ in $\mathrm{Ind}(\mathcal{D}) \hookrightarrow \widehat{\mathcal{D}}$ are compact because the functor

$$P \longmapsto \mathrm{Hom}(y(X), P) = P(X)$$

respects all colimits.
The same applies to the objects of $\mathrm{Kar}(\mathcal{D})$ because the restriction functor

$$\widehat{\mathrm{Kar}(\mathcal{D})} \longrightarrow \widehat{\mathcal{D}} \qquad \text{is an equivalence.}$$

- Consider an ind-object of $\mathcal{D}$

$$P = \varinjlim_{\mathcal{I}} y(X_i)$$

written as a filtering colimit of representable objects $\quad y(X_i)$ indexed by a small filtering category $\mathcal{I}$.
If P is a compact object, the identity morphism

$$P \xrightarrow{=} P = \varinjlim_{\mathcal{I}} y(X_i)$$

factorizes for an object i_0 of $\mathcal{I}$ in

$$P \xrightarrow{j} y(X_{i_0}) \xrightarrow{r} \varinjlim_{\mathcal{I}} y(X_i) = P\,.$$

So

$$j \circ r : y(X_{i_0}) \longrightarrow y(X_{i_0})$$

comes from an idempotent of $\mathcal{D}$

$$p : X_{i_0} \longrightarrow X_{i_0} \quad \text{verifying} \quad p \circ p = p\,,$$

and P is the image of the object (X_{i_0}, p) of $\mathrm{Kar}(\mathcal{C})$.

Application to a criterion of equivalence between toposes of presheaves:

Corollary. –
Let $\mathcal{C}$ and $\mathcal{D}$ be two essentially small categories.
Then the equivalences of presheaf toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \widehat{\mathcal{D}}$$

correspond to equivalences of categories

$$\mathrm{Kar}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Kar}(\mathcal{D})\,.$$

Remark. –
In particular, if $\mathcal{C}$ and $\mathcal{D}$ are Karoubi-complete,
the equivalences of presheaf toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \widehat{\mathcal{D}}$$

correspond to equivalences of categories

$$\mathcal{C} \xrightarrow{\sim} \mathcal{D}\,.$$

Application to the presentation of classifying toposes of presheaf type topos:

Corollary. – *Let $\mathbb{T}$ be a presheaf type theory.*
Let $\mathcal{M}$ be the category of finitely presentable models of $\mathbb{T}$.
Then:

- *The category $\mathcal{M}$ is essentially small.*
- *It is Karoubi-complete.*
- *We have a canonical equivalence of toposes* $[\mathcal{M}, \mathrm{Set}] = \widehat{\mathcal{M}^{\mathrm{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$.

Remark. – The universal model of $\mathbb{T}$ in $[\mathcal{M}, \mathrm{Set}]$ consists in associating

- to any sort A of the signature Σ of $\mathbb{T}$, the presheaf
$$M \longmapsto MA$$
- to any function symbol $f : A_1 \cdots A_n \to B$ of Σ, the presheaf morphism
$$M \longmapsto (MA_1 \times \cdots \times MA_n \xrightarrow{Mf} MB)\,,$$
- to any relation symbol $R \rightarrowtail A_1 \cdots A_n$ of Σ, the sub-presheaf
$$M \longmapsto (MR \hookrightarrow MA_1 \times \cdots \times MA_n)\,.$$

Syntactic characterization of presheaf type theories:

Theorem (Caramello). – *Let $\mathbb{T}$ be a geometric theory of signature Σ.*
Let $\mathcal{C}_{\mathbb{T}}$ be the geometric syntactic category of $\mathbb{T}$,
equipped with its syntactic topology $J_{\mathbb{T}}$.
Then the following conditions are equivalent:

(1) *The theory $\mathbb{T}$ is presheaf type.*

(2) *Any object of $\mathcal{C}_{\mathbb{T}}$, i.e. any geometric formula of Σ*

$$\varphi(\vec{x})$$

admits in $\mathcal{C}_{\mathbb{T}}$ a $J_{\mathbb{T}}$-covering

$$\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x})$$

by formulas $\varphi_i(\vec{x}_i)$ which are "$J_{\mathbb{T}}$-irreducible".

Remark. – A geometric formula $\psi(\vec{y})$ is "irreducible" if,
for any family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$\theta_j(\vec{y}_j, \vec{y}) : \psi_j(\vec{y}_j) \longrightarrow \psi(\vec{y}) \quad \text{such that} \quad \psi \vdash_{\vec{y}} \bigvee_j (\exists\, \vec{y}_j)\, \theta_j(\vec{y}_j, \vec{y})$$

is $\mathbb{T}$-provable, there exists an index j_0 such that the morphism

$$\theta_{j_0}(\vec{y}_{j_0}, \vec{y}) : \psi_{j_0}(\vec{y}_{j_0}) \longrightarrow \psi(\vec{y})$$

admits a section.

Presentation of finitely presentable models by irreducible formulas:

Corollary. – *Let $\mathbb{T}$ be a geometric theory of presheaf type. Let $\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$ be the full subcategory of $\mathcal{C}_{\mathbb{T}}$ consisting of irreducible geometric formulas. Then:*

(i) *The canonical functor*

$$\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}} \xrightarrow{\ell} \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$$

extends to an equivalence of toposes

$$\widehat{\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}\,.$$

(ii) *If $\mathcal{M}$ denotes the category of finitely presentable models of $\mathbb{T}$, we have an induced equivalence of categories*

$$\mathcal{M}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$$

which associates with any finitely presentable model

$$M$$

an irreducible geometric formula

$$\varphi_M$$

which "presents" the set-based model M.

The notion of presentation of a set-based model by a formula:

Definition. – *Let $\mathbb{T}$ be a geometric theory of presheaf type (or more generally whose set-based models are conservative). We say that a set-based model of $\mathbb{T}$*

$$M$$

is "presented" by a geometric formula

$$\varphi(\vec{x})$$

of context $\vec{x} = x_1^{A_1} \cdots x_k^{A_k}$ if, for all set-based model of $\mathbb{T}$

$$N\,,$$

considering a model morphism

$$M \longrightarrow N$$

is equivalent to considering a family of elements

$$n_1 \in NA_1, \cdots, n_k \in NA_k$$

which satisfies the condition

$$(n_1, \cdots, n_k) \in N\varphi(\vec{x}) \hookrightarrow NA_1 \times \cdots \times NA_k\,.$$

Remark. – We can also say that the model M is defined by k generators $x_1^{A_1}, \cdots, x_k^{A_k}$ and the relation $\varphi(\vec{x})$.

The notion of irreducible object of a topos or a site:

Definition. –

(i) *A object E of a topos $\mathcal{E}$ is said to be "irreducible" if, for any family of morphisms of $\mathcal{E}$*

$$E_i \longrightarrow E\,, \qquad i \in I\,,$$

such that $\coprod_i E_i \to E$ is an epimorphism, there exists an index $i_0 \in I$ such that the morphism

$$E_{i_0} \longrightarrow E$$

admits a section.

(ii) *An object X of an essentially small category $\mathcal{C}$ endowed with a Grothendieck topology J is said to be "J-irreducible" if the unique J-covering sieve of X is the maximal sieve.*

Relations between the notions of irreducibility:

- For any site $(\mathcal{C}, J)$, the canonical functor
$$\ell : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}_J$$
transforms any J-irreducible object of $\mathcal{C}$ into an irreducible object of the topos $\widehat{\mathcal{C}}_J$.
- Conversely, if the topology J of $\mathcal{C}$ is subcanonical, any object of $\mathcal{C}$ that the functor
$$\ell : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}_J$$
transforms into an irreducible object of the topos $\widehat{\mathcal{C}}_J$ is a J-irreducible object of $\mathcal{C}$.
- In particular, for any geometric theory $\mathbb{T}$ and its syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, a geometric formula
$$\varphi(\vec{x}) \qquad (= \text{object of } \mathcal{C}_{\mathbb{T}})$$
is irreducible if and only if its image by the functor
$$\ell : \mathcal{C}_{\mathbb{T}} \longrightarrow \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$$
is an irreducible object of the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$.

Proof in one direction of the theorem and its corollary by Grothendieck's "comparison lemma":

- Let $\mathbb{T}$ be a geometric theory.
  Let $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ be its geometric syntactic site and $\mathcal{C}_{\mathbb{T}}^{\text{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}}$ the full subcategory of $\mathcal{C}_{\mathbb{T}}$ consisting of geometric formulas $\varphi(\vec{x})$ which are "$J_{\mathbb{T}}$-irreducible".
- Requiring that any geometric formula admits a $J_{\mathbb{T}}$-covering by irreducible formulas amounts to requiring that the full sub-category

$$\mathcal{C}_{\mathbb{T}}^{\text{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}}$$

  be $J_{\mathbb{T}}$-dense.
- In this case, the topology $J_{\mathbb{T}}$ of $\mathcal{C}_{\mathbb{T}}$ induces on $\mathcal{C}_{\mathbb{T}}^{\text{ir}}$ the discrete topology, and Grothendieck's "comparison lemma" yields an equivalence of toposes

$$\widehat{\mathcal{C}_{\mathbb{T}}^{\text{ir}}} \xrightarrow{\sim} \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}\,.$$

Proof of the reverse direction of the theorem and its corollary by a topos-theoretic bridge:

- Consider a geometric theory $\mathbb{T}$
  assumed to be "presheaf type".
  We already know that the category of its finitely presentable models

$$\mathcal{M}$$

  is "Karoubi-complete" and defines an equivalence

$$\widehat{\mathcal{M}^{\text{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} .$$

- We are going to calculate the invariant of toposes

$$\mathcal{E} \longmapsto \left\{ \begin{array}{c} \text{full subcategory of } \mathcal{E} \\ \text{made up of irreducible objects} \end{array} \right\}$$

  on the two sides of the equivalence of toposes

$$\widehat{\mathcal{M}^{\text{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} .$$

Calculation of irreducible objects of a topos:

Lemma. – *Let* $(\mathcal{C}, J)$ *be a site equipped with the canonical functor* $\ell : \mathcal{C} \to \widehat{\mathcal{C}}_J = \mathcal{E}$.

(i) *Any irreducible object* E *of the topos* $\widehat{\mathcal{C}}_J = \mathcal{E}$ *is a "retract" of the image* $\ell(X)$ *of an object* X *of* $\mathcal{C}$ *in the sense that there exists an idempotent*

$$p : \ell(X) \longrightarrow \ell(X) \qquad \text{verifying} \qquad p \circ p = p$$

such that $E = \ker(\ell(X) \overset{p}{\underset{\mathrm{id}}{\rightrightarrows}} \ell(X))$.

(ii) *If the topology* J *of* $\mathcal{C}$ *is subcanonical, and the category* $\mathcal{C}$ *is Karoubi-complete, the canonical functor* ℓ *induces an equivalence*

$$\ell : \mathcal{C}^{\mathrm{ir}} \xrightarrow{\sim} \mathcal{E}^{\mathrm{ir}}$$

from the full subcategory $\mathcal{C}^{\mathrm{ir}}$ *of* $\mathcal{C}$ *of* J*-irreducible objects onto the full subcategory* $\mathcal{E}^{\mathrm{ir}}$ *of irreducible objects of the topos* $\mathcal{E}$.

Proof of the formula for calculating irreducible objects:

- For any object E of a sheaf topos $\mathcal{E} = \widehat{\mathcal{C}}_J$, there exists a family of objects X_i of $\mathcal{C}$ and morphisms of $\mathcal{E}$

$$\ell(X_i) \longrightarrow E$$

  such that the morphism $\coprod \ell(X_i) \to E$ is an epimorphism.

- If E is an irreducible object, there exist an index i_0 and morphisms of $\mathcal{E}$

$$E \xrightarrow{j} \ell(X_{i_0}) \xrightarrow{r} E$$

  such that $r \circ j = \mathrm{id}_E$.
  Putting $p = j \circ r$, we have $p \circ p = p$ and

$$E = \ker(\ell(X_{i_0}) \underset{\mathrm{id}}{\overset{p}{\rightrightarrows}} \ell(X_{i_0}))\,.$$

- If $\mathcal{C}$ is Karoubi-complete and J is subcanonical, we get an equivalence of categories

$$\mathcal{C}^{\mathrm{ir}} \xrightarrow{\sim} \mathcal{E}^{\mathrm{ir}}$$

  since, as we have already seen,

  an object X of $\mathcal{C}$ is J-irreducible
  if and only if $\ell(X)$ is irreducible in the topos $\widehat{\mathcal{C}}_J = \mathcal{E}$.

End of the proof of the theorem and its corollary:

- We consider a geometric theory $\mathbb{T}$ of presheaf type,
  its category $\mathcal{M}$ of finitely presentable models
  and the canonical equivalence
  $$\widehat{\mathcal{M}^{\text{op}}} \longrightarrow \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} .$$
- The category $\mathcal{M}$ is Karoubi-complete
  and any object of $\mathcal{M}$ is irreducible for the discrete topology,
  so we have an induced equivalence of categories
  $$\mathcal{M}^{\text{op}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}^{\text{ir}} .$$
- The category $\mathcal{C}_{\mathbb{T}}$ is Karoubi-complete (because it is cartesian),
  and the topology $J_{\mathbb{T}}$ is subcanonical,
  so we also have an equivalence of categories
  $$\mathcal{C}_{\mathbb{T}}^{\text{ir}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}^{\text{ir}} .$$
- So we have a canonical equivalence $\quad \mathcal{M}^{\text{op}} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}}^{\text{ir}}$
  and the full subcategory $\quad \mathcal{C}_{\mathbb{T}}^{\text{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}}$
  is dense for the syntactic topology $J_{\mathbb{T}}$.

Characterization of presheaf type theories by a triple correspondence between syntax and semantics:

Theorem (Caramello). – *Let $\mathbb{T}$ be a fist-order theory of signature Σ. Then $\mathbb{T}$ is presheaf type if and only if it satisfies the following three conditions:*

(1) *The finitely presentable set-based models of $\mathbb{T}$ are conservative, in the sense that an implication property between geometric formulas of Σ*

$$\varphi \vdash_{\vec{x}} \psi$$

is $\mathbb{T}$-provable if (and only if) it is verified by all finitely presentable models of $\mathbb{T}$.

(2) *Any finitely presentable set-based model M of $\mathbb{T}$ is "presented" by a geometric formula of Σ*

$$\varphi_M(\vec{x}) \quad \text{in a context} \quad \vec{x} = x_1^{A_1} \cdots x_k^{A_k} \,,$$

in the sense that for any set-based model N of $\mathbb{T}$, considering a model morphism

$$M \longrightarrow N$$

is equivalent to considering a family of elements

$$(n_1, \cdots, n_k) \in N\varphi_M(\vec{x}) \hookrightarrow NA_1 \times \cdots \times NA_k \,.$$

(3) *For any sequence of sorts $A_1, \cdots, A_n$ of Σ and any family of subsets*

$$P_M \hookrightarrow MA_1 \times \cdots \times MA_n$$

indexed by finitely presentable set-based models of $\mathbb{T}$ which is "functorial" in the sense that for any model morphism

$$M \longrightarrow N$$

the induced map

$$MA_1 \times \cdots \times MA_n \longrightarrow NA_1 \times \cdots \times NA_n$$

sends the subset P_M into the subset P_N, there exists a geometric formula of Σ

$$\varphi(\vec{x}) \quad \text{in a context} \quad \vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

which defines the functorial family $M \mapsto (P_M \hookrightarrow MA_1 \times \cdots \times MA_n)$, in the sense that for any finitely presentable set-based model M of $\mathbb{T}$

$$P_M = M\varphi(\vec{x}) \hookrightarrow MA_1 \times \cdots \times MA_n\,.$$

Why finitely presentable set-based models of a presheaf type theory are conservative:

We have shown this property by the "topos-theoretic bridge" which consists in computing the invariant

$$\text{topos } \mathcal{E} \longmapsto \left\{ \begin{array}{c} \text{full subcategory of } \mathrm{pt}(\mathcal{E}) \\ \text{made up of compact objects} \end{array} \right\}$$

on the two sides of an equivalence of toposes

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} .$$

We thus obtain an equivalence of categories

$\mathrm{Kar}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathcal{M} =$ category of finitely presentable set-based models,

and therefore an equivalence of topos $\widehat{\mathcal{M}^{\mathrm{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$.

Via this equivalence, the interpretations in the universal model of $\mathbb{T}$ of geometric formulas

$$\varphi(\vec{x}), \psi(\vec{x}) \quad \text{in a context} \quad \vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

are the sub-presheaves

$$M \longmapsto \left\{ \begin{array}{lcl} M\varphi(\vec{x}) & \hookrightarrow & MA_1 \times \cdots \times MA_n \,, \\ M\psi(\vec{x}) & \hookrightarrow & MA_1 \times \cdots \times MA_n \,. \end{array} \right.$$

So $\varphi \vdash_{\vec{x}} \psi$ is $\mathbb{T}$-provable if and only if it is verified by all finitely presentable models M.

Why finitely presentable set-based models of a presheaf type theory are presented by formulas:

This property was shown by the "topos-theoretic bridge" which consists in calculating the invariant of toposes

$$\mathcal{E} \longmapsto \left\{ \begin{array}{c} \text{full subcategory of } \mathcal{E} \\ \text{consisting of irreducible objects} \end{array} \right\}$$

on the two sides of the equivalence of toposes

$$\widehat{\mathcal{M}^{\mathrm{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} .$$

Indeed, we obtain in this way an equivalence of categories

$$\mathcal{M}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$$

which associates to any finitely presentable set-based model M of $\mathbb{T}$ an (irreducible) geometric formula

$$\varphi_M$$

which "presents" the model M.

Why are the functorial properties of families of elements of finitely presentable models of a presheaf type theory defined by geometric formulas :

We prove this property by the "topos-theoretic bridge" which consists in computing the invariant

$$\left\{\begin{array}{c}\text{topos } \mathcal{E} \text{ endowed} \\ \text{with a model } U \text{ of } \mathbb{T}\end{array}\right\} \longmapsto \left\{\begin{array}{c}\text{set of subobjects of the object of } \mathcal{E} \\ U\top(x_1^{A_1}\cdots x_n^{A_n})\end{array}\right\}$$

on both sides of the equivalence of toposes endowed with the universal model of $\mathbb{T}$

$$\widehat{\mathcal{M}^{\mathrm{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}\,.$$

We obtain on the left-hand side the set of sub-presheaves

$$M \longmapsto (P_M \hookrightarrow MA_1 \times \cdots \times MA_n)$$

and on the right-hand side the set of classes of geometric formulas

$$\varphi(\vec{x}) \hookrightarrow \top(\vec{x}) = \top(x_1^{A_1}\cdots x_n^{A_n})\,.$$

How to show that a theory is of presheaf type if it satisfies the three conditions:

- We consider a geometric theory $\mathbb{T}$ of signature Σ which satisfies the conditions (1), (2), (3).
- We consider

$$\mathcal{C}_{\mathbb{T}} = \text{geometric syntactic category of } \mathbb{T},$$
$$J_{\mathbb{T}} = \text{syntactic topology of } \mathbb{T},$$
$$\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} = \text{full subcategory of } \mathcal{C}_{\mathbb{T}} \text{ consisting of irreducible geometric formulas.}$$

- In order to show that $\mathbb{T}$ is of presheaf type, it suffices to establish that $\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$ is dense in $\mathcal{C}_{\mathbb{T}}$ for the topology $J_{\mathbb{T}}$.

From syntax to semantics, via the interpretations of formulas:

- Let $\mathcal{M}$ = category of finitely presentable set-based models of $\mathbb{T}$
  = full subcategory of $\mathbb{T}\text{-mod}\,(\mathrm{Set})$
  consisting of compact objects.
- We have the interpretation functor

$$\begin{array}{rcl} I : \mathcal{C}_{\mathbb{T}} & \longrightarrow & \widehat{\mathcal{M}^{\mathrm{op}}} = [\mathcal{M}, \mathrm{Set}]\,, \\ \text{formula } \varphi(\vec{x}) & \longmapsto & \text{presheaf of interpretations} \\ & & \quad M \mapsto M\varphi(\vec{x})\,, \end{array}$$

$$\left.\begin{array}{c} \mathbb{T}\text{-provably} \\ \text{functional} \\ \text{formula} \\ \theta(\vec{x},\vec{y}) : \varphi(\vec{x}) \to \psi(\vec{y}) \end{array}\right\} \longmapsto \left\{\begin{array}{c} \text{presheaf morphism} \\ M \mapsto (M\varphi(\vec{x}) \to M\psi(\vec{y})) \\ \text{consisting of maps whose graphs} \\ \text{are} \quad M\theta(\vec{x},\vec{y}) \hookrightarrow M\varphi(\vec{x}) \times M\psi(\vec{y}). \end{array}\right.$$

- It follows from properties (1) and (3) that this functor

$$I : \mathcal{C}_{\mathbb{T}} \longrightarrow \widehat{\mathcal{M}^{\mathrm{op}}}$$

is fully faithful.

Irreducibility of presentation formulas of finitely presentable models:

- It follows from (2) that any finitely presentable model M, object of
$$\mathcal{M}^{\rm op} \hookrightarrow \widehat{\mathcal{M}^{\rm op}}\,,$$
is the image of a formula φ_M, object of $\mathcal{C}_{\mathbb{T}}$, by the functor
$$I : \mathcal{C}_{\mathbb{T}} \longrightarrow \widehat{\mathcal{M}^{\rm op}}\,.$$
- Consider a $J_{\mathbb{T}}$-covering of φ_M in $\mathcal{C}_{\mathbb{T}}$
$$\theta_i(\vec{x}_i,\vec{x}) : \varphi_i(\vec{x}_i) = \varphi_i \longrightarrow \varphi_M = \varphi_M(\vec{x})\,.$$
By definition of $J_{\mathbb{T}}$, the implication
$$\varphi_M \vdash_{\vec{x}} \bigvee_i (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i,\vec{x}) \qquad \text{is } \mathbb{T}\text{-provable.}$$
So the presheaf morphism in $\widehat{\mathcal{M}^{\rm op}} = [\mathcal{M}, \mathrm{Set}]$
$$\coprod_i I(\varphi_i) \longrightarrow I(\varphi_M) = y(M) = \mathrm{Hom}(M,\bullet)$$
is an epimorphism, and there exists an index i_0 such that
$$\mathrm{id}_M \in \mathrm{Hom}(M,M) \qquad \text{is the image of an element of } I(\varphi_{i_0}).$$
- By full faithfulness of the functor $I : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{M}^{\rm op}}$,
this means that the morphism of $\mathcal{C}_{\mathbb{T}}$
$$\theta_{i_0}(\vec{x}_{i_0},\vec{x}) : \varphi_{i_0}(\vec{x}_{i_0}) \longrightarrow \varphi_M(\vec{x})$$
is split.

Density of irreducible formulas:

- Consider a geometric formula

$$\varphi = \varphi(\vec{x}) = \text{object of } \mathcal{C}_{\mathbb{T}} \,.$$

- There exists in $\widehat{\mathcal{M}^{\mathrm{op}}} = [\mathcal{M}, \mathrm{Set}]$ a family of morphisms

$$y(M_i) \longrightarrow I(\varphi)$$

such that

$$\coprod_i y(M_i) \longrightarrow I(\varphi)$$

is an epimorphism.

- Each $y(M_i) \to I(\varphi)$ is the image of a morphism of $\mathcal{C}_{\mathbb{T}}$

$$\theta_i(\vec{x}_i, \vec{x}) : \varphi_{M_i}(\vec{x}_i) = \varphi_{M_i} \longrightarrow \varphi = \varphi(\vec{x}) \,,$$

and the implication

$$\varphi \vdash_{\vec{x}} \bigvee_i (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x})$$

is verified in any finitely presentable model M,
so is $\mathbb{T}$-provable.

- So $\varphi = \varphi(\vec{x})$ admits a $J_{\mathbb{T}}$-covering by the formulas

$$\varphi_{M_i} = \varphi_{M_i}(\vec{x}_i)$$

which are $J_{\mathbb{T}}$-irreducible.

A counterexample: the theory of fields.

Corollary. –
The theory of fields [resp. of commutative fields] can be formalized as a coherent theory but it is not of presheaf type.

Proof. –

- The theory of fields [resp. commutative fields] is the quotient theory of the (algebraic) theory of rings [resp. of commutative rings] defined by adding the coherent axiom

$$\top \vdash_k k = 0 \vee (\exists\, k')(k \cdot k' = 1 \wedge k' \cdot k = 1)\,.$$

- The property (without free variable) of fields K

$$\text{“char}(K) = 0\text{”}$$

is functorial, but it is not defined by any geometric formula.