II. Provability, quotient theories and corresponding topologies (presentation by L. Lafforgue)

Reminder of the uses already made of the notion of provability:

- **To define morphisms of syntactic categories:**
  These are the formulas

  $$\varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{y})} \psi(\vec{y})$$

  which are "$\mathbb{T}$-provably functional" in the sense that

  $$\left.\begin{array}{l} \theta \vdash_{\vec{x},\vec{y}} \varphi \wedge \theta \\ \varphi \vdash_{\vec{x}} (\exists\, \vec{y})\, \theta(\vec{x},\vec{y}) \\ \theta(\vec{x},\vec{y}) \wedge \theta(\vec{x},\vec{y}\,') \vdash_{\vec{x},\vec{y},\vec{y}\,'} \vec{y} = \vec{y}\,' \end{array}\right\} \begin{array}{c} \text{are provable} \\ \text{in the theory } \mathbb{T} \text{ under consideration.} \end{array}$$

- **To define the objects of Cartesian syntactic categories:**
  These are the formulas of the form

  $$\varphi(\vec{x}) = (\exists\, \vec{y})\, \psi(\vec{x},\vec{y})$$

  where ψ is a "Horn formula" such that

  $$\psi(\vec{x},\vec{y}) \wedge \psi(\vec{x},\vec{y}\,') \vdash_{\vec{x},\vec{y},\vec{y}\,'} \vec{y} = \vec{y}\,'$$

  is provable in the theory $\mathbb{T}$ under consideration.

- **To define the notion of quotient theory:**
  A (first-order geometric) theory $\mathbb{T}'$
  is a "quotient" of a theory $\mathbb{T}$
  if it has the same signature
  and if any axiom of $\mathbb{T}$ is provable
  from the axioms of $\mathbb{T}'$.

- **To define the notion of syntactic equivalence:**
  Two (first-order geometric) theories
  with the same signature
  are said to be "syntactically equivalent"
  if each is a quotient of the other,
  that is, if any axiom of one
  is provable from the axioms of the other.

What does "provable" mean?

Note. –

So far we have used
the notion of "provable"
without specifying its meaning.

Definition. –

Let Σ be a signature.
Let $\mathbb{T}$ be a first-order geometric theory of signature Σ, defined by a family of axioms

$$\varphi_i \vdash \psi_i \,, \quad i \in I \,.$$

Then a property linking geometric formulas φ, ψ of Σ

$$\varphi \vdash \psi$$

is said to be "provable" in $\mathbb{T}$ or "$\mathbb{T}$-provable"
if it can be deduced from the axioms of $\mathbb{T}$
by the "inference rules of geometric logic".

The essential characteristics of inference rules:

- The "rules of inference" of geometric logic are common to all first-order geometric theories.
- They are such that, for any signature Σ and for any Σ-structure M in a topos $\mathcal{E}$ which satisfies a family of axioms

$$\varphi_i \vdash \psi_i\,, \quad i \in I\,,$$

then M also satisfies any property

$$\varphi \vdash \psi$$

which is deduced from these axioms by the "rules of inference".

- Conversely, if $\mathbb{T}$ is a geometric theory of signature Σ, defined by axioms $\varphi_i \vdash \psi_i$, $i \in I$, then the "universal model" $M_{\mathbb{T}}$ of $\mathbb{T}$ in the classifying topos $\mathcal{E}_{\mathbb{T}}$ satisfies a property $\varphi \vdash \psi$ only if it follows from the axioms by the rules of inference.

The exhaustive list of inference rules of geometric logic:

(1) The cut rule. –

Two properties of the form

$$\varphi_1 \vdash_{\vec{x}} \varphi_2 \qquad \textit{and} \qquad \varphi_2 \vdash_{\vec{x}} \varphi_3$$

imply the property

$$\varphi_1 \vdash_{\vec{x}} \varphi_3 \,.$$

Verification. –

This rule is valid in any topos $\mathcal{E}$
because if three subobjects

$$E_1, E_2, E_3 \quad \text{of an object } E \text{ of } \mathcal{E}$$

satisfy the inclusion relations

$$E_1 \subseteq E_2 \qquad \text{and} \qquad E_2 \subseteq E_3 \,,$$

then they also satisfy

$$E_1 \subseteq E_3 \,.$$

(2) The rule of identity. –

For any term f, the property

$$\top \vdash_{\vec{x}} f = f$$

is an implicit axiom of any theory.

Verification. –

This rule is valid in any topos $\mathcal{E}$,
because for any morphism of $\mathcal{E}$

$$f : E \longrightarrow E' ,$$

the fiber product associated with the diagram

$$\begin{array}{ccc} & & E \\ & & \big\downarrow {\scriptstyle f \times f} \\ E' & \stackrel{\Delta}{\hookrightarrow} & E' \times E' \end{array}$$

is the total subobject E of E.

(3) The rules of equality. –

- *A property of the form* $\top \vdash_{\vec{x}} f_1 = f_2$ *is equivalent to the property* $\top \vdash_{\vec{x}} f_2 = f_1$.
- *Two properties of the form*

$$\top \vdash_{\vec{x}} f_1 = f_2 \qquad \text{and} \qquad \top \vdash_{\vec{x}} f_2 = f_3$$

imply the property

$$\top \vdash_{x} f_1 = f_3 .$$

Verification. –
These rules are valid in any topos $\mathcal{E}$
because, for all morphisms of $\mathcal{E}$

$$E \xrightarrow{f_1} E' , \quad E \xrightarrow{f_2} E' \qquad [\text{resp. and } E \xrightarrow{f_3} E']$$

the equality between morphisms $f_1 = f_2$ is equivalent to the equality $f_2 = f_1$,
and the equalities of morphisms

$$f_1 = f_2 \qquad \text{and} \qquad f_2 = f_3$$

imply the equality

$$f_1 = f_3 .$$

(4) Substitution rules. –

- *If f_1, f_2 are two terms with the same context $\vec{x}$, and f_1', f_2' are two terms deduced from f_1, f_2 by substitution of a term f for a variable [resp. deduced from a term f by substitution of f_1 and f_2 for a variable], then the property*

$$\top \vdash f_1 = f_2$$

implies the property

$$\top \vdash f_1' = f_2' .$$

- *If f_1, f_2 are two terms with the same context $\vec{x}$, if R is a relation and if R_1, R_2 are the two relations deduced from R by substitution of f_1 and f_2 for a variable, then the property*

$$\top \vdash_{\vec{x}} f_1 = f_2$$

implies the properties

$$R_1 \vdash R_2 \quad \text{and} \quad R_2 \vdash R_1 \quad (\text{denoted by } R_1 \dashv\vdash R_2).$$

Verification. –

- The first of these rules is valid in any topos $\mathcal{E}$ because, for all morphisms of $\mathcal{E}$

  $$E \xrightarrow{f_1} E', \quad E \xrightarrow{f_2} E'$$

  and

  $$E_0 \xrightarrow{f} E \qquad [\text{resp.} \quad E' \xrightarrow{f} E_0'],$$

  the equality between morphisms

  $$f_1 = f_2$$

  implies the equality

  $$f_1 \circ f = f_2 \circ f \qquad [\text{resp.} \quad f \circ f_1 = f \circ f_2].$$

- The second of these rules is valid in any topos $\mathcal{E}$ because, for all morphisms of $\mathcal{E}$

  $$E \xrightarrow{f_1} E', \quad E \xrightarrow{f_2} E'$$

  and for any subobject

  $$R \hookrightarrow E'$$

  the equality between morphisms

  $$f_1 = f_2$$

  implies the equality of the pull-back subobjects

  $$f_1^{-1} R = f_2^{-1} R \qquad \text{in object } E.$$

(5) The rules of finitary conjunctions. –

- *For any formula* φ *in a context* $\vec{x}$*, the property*

$$\varphi \vdash_{\vec{x}} \top$$

is an implicit axiom of any theory.

- *For any finite family* $\varphi_1, \cdots, \varphi_k$ *of formulas with the same context* $\vec{x}$ *and for any formula* φ *of context* $\vec{x}$*, the property*

$$\varphi \vdash_{\vec{x}} \varphi_1 \wedge \cdots \wedge \varphi_k$$

is equivalent to the family of properties

$$\varphi \vdash_x \varphi_i \,, \qquad 1 \leq i \leq k\,.$$

Verification. –
These rules are valid in any topos $\mathcal{E}$ because,
for any subobject E' of an object E of $\mathcal{E}$, we have

- E' is contained in the total subobject E of E,
- E' is contained in subobjects $E_1, \cdots, E_k$ of E if and only if it is contained in their intersection $E_1 \wedge \cdots \wedge E_k$.

(6) The rules of disjunctions. –

- *For any formula* φ *in a context* $\vec{x}$*, the property*

$$\perp \vdash_{\vec{x}} \varphi$$

is an <u>*implicit axiom*</u> *of any theory.*

- *For any family of formulas* $(\varphi_i)_{i \in I}$ *in the same context* $\vec{x}$*, and for any formula* φ *of context* $\vec{x}$*, the property*

$$\bigvee_{i \in I} \varphi_i \vdash_{\vec{x}} \varphi$$

is <u>*equivalent*</u> *to the family of properties*

$$\varphi_i \vdash_x \varphi \,, \qquad i \in I \,.$$

Verification. –
These rules are valid <u>in any topos</u> $\mathcal{E}$ because,
for any subobject E' of an object E of $\mathcal{E}$, we have

- E' contains the empty subobject $\emptyset$ of E,
- E' contains subobjects E_i, $i \in I$, of E if and only if it contains their <u>union</u> $\bigvee_{i \in I} E_i$.

(7) The distributivity rule. –

For any formulas φ and φ_i, $i \in I$, with the same context $\vec{x}$, the equivalence

$$\varphi \wedge \bigvee_{i \in I} \varphi_i \dashv\vdash_{\vec{x}} \bigvee_{i \in I} \varphi \wedge \varphi_i$$

is an implicit axiom of any theory.

Note. –

The reverse part of this equivalence

$$\bigvee_{i \in I} \varphi \wedge \varphi_i \vdash_{\vec{x}} \varphi \wedge \bigvee_{i \in I} \varphi_i$$

follows from (5) and (6).

Verification. –

This rule is valid in any topos $\mathcal{E}$ because, for any subobject E' of an object E of $\mathcal{E}$, the intersection functor with E' in E

$$E' \wedge \bullet = E' \times_E \bullet$$

respects both limits and colimits, so also unions of subobjects.

(8) The rule of existential quantification. –
For any disjoint contexts $\vec{x}$ and $\vec{y}$,
any formula φ of context $\vec{x}, \vec{y}$
and any formula ψ of context $\vec{x}$,
the property

$$\varphi \vdash_{\vec{x},\vec{y}} \psi$$

is equivalent to the property

$$(\exists\,\vec{y})\,\varphi \vdash_{\vec{x}} \psi\,.$$

Verification. –
This rule is valid in any topos $\mathcal{E}$ because,
for any morphism of $\mathcal{E}$

$$p : E' \longrightarrow E$$

and for any subobjects

$$E_0 \hookrightarrow E \quad \text{and} \quad E_0' \hookrightarrow E' \,,$$

the inclusion relations between subobjects

$$E_0' \subseteq p^{-1} E_0 = E' \times_E E_0 \quad \text{in } E'$$

and

$$\mathrm{Im}(E_0' \hookrightarrow E' \xrightarrow{p} E) \subseteq E_0 \quad \text{in } E$$

are equivalent.

(9) The Frobenius rule. –

For any formula φ *of context* $\vec{x}, \vec{y}$
and any formula ψ *of context* $\vec{x}$*, as in (8),*
the equivalence

$$(\exists\, \vec{y})\ \varphi \wedge \psi \dashv\vdash_{\vec{x}} (\exists\, \vec{y})(\varphi \wedge \psi)$$

is an implicit axiom of any theory.

Note. –

The reverse part of this equivalence

$$(\exists\, \vec{y})(\varphi \wedge \psi) \vdash_{\vec{x}} (\exists\, \vec{y})\ \varphi \wedge \psi$$

follows from (5) and (8).

Verification. –

This rule is valid in any topos $\mathcal{E}$ because,
for any morphism of $\mathcal{E}$

$$p : E' \longrightarrow E$$

and for any subobject $E_0 \hookrightarrow E$,
the fiber product functor

$$E_0 \times_E \bullet$$

respects both limits and colimits,
therefore also the images by the morphism $p : E' \to E$.

Geometric logic and its fragments:

Definition. –

(i) *We call geometric logic (of first order) the list of rules of inference (1) to (9) from the previous pages.*

(ii) *We call coherent fragment of this logic the list deduced from the previous one by limiting rules (6) and (7) at the finitary disjunctions* $\varphi_1 \vee \cdots \vee \varphi_k$.

(iii) *We call regular fragment of this logic the list deduced from the previous one forgetting rules (6) and (7).*

Remark. –
If $\mathbb{T}$ is a coherent [resp. regular] theory,
then a property linking coherent [resp. regular] formulas

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}$ in the sense of geometric logic
if and only if it is in the sense of coherent logic [resp. regular logic].

The semantic expression of provability:

Theorem. –

Let $\mathbb{T}$ be a geometric theory [resp. coherent, resp. regular theory] of signature Σ.

Then a property linking geometric formulas [resp. coherent, resp. regular formulas] of Σ

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}$

in the sense of geometric logic [resp. coherent, resp. regular logic]

if and only if it is verified

by any model M of $\mathbb{T}$

in any topos $\mathcal{E}$.

Remark. –

This theorem implies the previous remark.

Partial proof:

Direct sense:

This results from the verifications made following the statements of the rules of inference of geometric logic.

Reverse direction:

Let $\mathcal{C}_{\mathbb{T}}$ be the geometric syntactic category [resp. coherent, resp. regular syntactic category] of $\mathbb{T}$, endowed with its syntactic topology $J_{\mathbb{T}}$.
Then the conclusion follows from the following facts:

- It suffices to prove that a property

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}$ if and only if it is verified by the universal model $M_{\mathbb{T}}$ of $\mathbb{T}$ in

$$\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})_{J_{\mathbb{T}}}} \, .$$

- Such a property $\varphi \vdash_{\vec{x}} \psi$
  is provable in $\mathbb{T}$
  if and only if, in the category $\mathcal{C}_{\mathbb{T}}$, the two subobjects

$$\varphi(\vec{x}) \hookrightarrow \top(\vec{x}) \quad \text{and} \quad \psi(\vec{x}) \hookrightarrow \top(\vec{x})$$

  satisfy the inclusion relation

$$\varphi(\vec{x}) \subseteq \psi(\vec{x})$$

  that is, if and only if the monomorphism

$$(\varphi \wedge \psi)(\vec{x}) \hookrightarrow \varphi(\vec{x})$$

  is an isomorphism.
- The syntactic topology $J_{\mathbb{T}}$ of $\mathcal{C}_{\mathbb{T}}$ is subcanonical.
  In other words, the canonical functor

$$\ell : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

  is fully faithful.
  In particular, a morphism of $\mathcal{C}_{\mathbb{T}}$ is an isomorphism
  if and only if
  its image by ℓ is an isomorphism.

Completeness or incompleteness of set-based models:

Question. –
For a geometric property

$$\varphi \vdash_{\vec{x}} \psi$$

to be provable in a theory $\mathbb{T}$,
is it enough that it is verified by set-based models of $\mathbb{T}$?

Answer. –

- Not in general:
  Many non-trivial topos have no points.
- Yes if $\mathbb{T}$ is a coherent theory,
  and if we suppose that the category of sets

$$\mathrm{Set}$$

  satisfies "the axiom of choice" (which is not constructive):
  "Any epimorphism of Set admits a section."

  This is the "completeness theorem" of Gödel.

Semantics of quotient theories:

Lemma. – *Let $\mathbb{T}$ be a geometric theory, $\mathbb{T}'$ a quotient theory of $\mathbb{T}$. Then:*

- **(i)** *The syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ is sent canonically to the syntactic category $\mathcal{C}_{\mathbb{T}'}$ of $\mathbb{T}'$. It has the same objects.*
- **(ii)** *For any topos $\mathcal{E}$,*
$$\mathbb{T}'\text{-mod}(\mathcal{E})$$
is a full subcategory of
$$\mathbb{T}\text{-mod}(\mathcal{E})\,.$$
- **(iii)** *The embeddings*
$$\mathbb{T}'\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E})$$
define a topos morphism
$$\mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}}$$
whose pull-back component extends the canonical functor
$$\mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{T}'}\,.$$

For the proof. – (i), (ii) and (iii) are consequences of the fact that any property provable in $\mathbb{T}$ is provable in $\mathbb{T}'$.

The "duality theorem" between quotient theories and subtoposes:

Theorem. –

Let $\mathbb{T}$ be a geometric theory.
Then:

(i) *For any quotient theory $\mathbb{T}'$ of $\mathbb{T}$, the associated topos morphism*

$$\mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is an embedding.

(ii) *The map*

$$\mathbb{T}' \longmapsto (\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}})$$

defines a bijection between

- *the set of equivalence classes of quotient theories $\mathbb{T}'$ of $\mathbb{T}$,*
- *the set of subtoposes of $\mathcal{E}_{\mathbb{T}}$.*

For the proof of this duality theorem:

Let $\mathcal{C}_{\mathbb{T}}$ be the geometric syntactic category of $\mathbb{T}$,
$J_{\mathbb{T}}$ its syntactic topology.

It is enough to show:

Proposition. –

(i) *For any quotient theory $\mathbb{T}'$ of $\mathbb{T}$, there exists a topology $J_{\mathbb{T}'}$ of $\mathcal{C}_{\mathbb{T}}$ containing $J_{\mathbb{T}}$ such that the morphism $\mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$ induces an isomorphism*

$$\mathcal{E}_{\mathbb{T}'} \xrightarrow{\sim} \widehat{(\mathcal{C}_{\mathbb{T}})_{J_{\mathbb{T}'}}} .$$

(ii) *The map*

$$\mathbb{T}' \longmapsto J_{\mathbb{T}'}$$

defines a bijection between

- *the set of equivalence classes of quotient theories $\mathbb{T}'$ of $\mathbb{T}$,*
- *the set of topologies J of $\mathcal{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.*

Constructive description of the correspondence between quotient theories and topologies:

The two applications in opposite directions are constructed as follows:

Definition. –

(i) *We associate to any quotient theory $\mathbb{T}'$ of $\mathbb{T}$ the topology on $\mathcal{C}_{\mathbb{T}}$*

$$J_{\mathbb{T}'} \supseteq J_{\mathbb{T}}$$

generated by $J_{\mathbb{T}}$ and the coverings

$$(\varphi \wedge \psi)(\vec{x}) \hookrightarrow \varphi(\vec{x})$$

indexed by the axioms of $\mathbb{T}'$

$$\varphi \vdash_{\vec{x}} \psi$$

which are not axioms of $\mathbb{T}$.

(ii) *We associate to any topology J of $\mathcal{C}_{\mathbb{T}}$ containing $J_{\mathbb{T}}$ the quotient theory $\mathbb{T}_J$ of $\mathbb{T}$ defined by the axioms of $\mathbb{T}$ completed with the axioms*

$$\varphi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x})$$

indexed by the J-covering families of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x}))_{i \in I}\,.$$

Match check:

These are the two parts of the following lemma:

Lemma. –

(i) *For any quotient theory $\mathbb{T}'$ of $\mathbb{T}$, the theory*

$$\mathbb{T}_{J_{\mathbb{T}'}}$$

associated with the topology $J_{\mathbb{T}'} \supseteq J_{\mathbb{T}}$ defined by $\mathbb{T}'$ is equivalent to $\mathbb{T}'$.

(ii) *For any topology J of $\mathcal{C}_{\mathbb{T}}$ containing $J_{\mathbb{T}}$, the topology*

$$J_{\mathbb{T}_J}$$

defined by the quotient theory $\mathbb{T}_J$ of $\mathbb{T}$ associated with J is equal to J.

For the proof. – We have to prove

for (i) that $\begin{cases} \mathbb{T}_{J_{\mathbb{T}'}} \text{ is a quotient of } \mathbb{T}', \\ \mathbb{T}' \text{ is a quotient of } \mathbb{T}_{J_{\mathbb{T}'}}, \end{cases}$

for (ii) that $\begin{cases} J \subseteq J_{\mathbb{T}_J}\,, \\ J_{\mathbb{T}_J} \subseteq J\,. \end{cases}$

Verification of the first part of (i): any axiom of $\mathbb{T}'$ is provable in $\mathbb{T}_{J_{\mathbb{T}'}}$.

Consider an axiom of $\mathbb{T}'$

$$\varphi \vdash_{\vec{x}} \psi \, .$$

Then the monomorphism of $\mathcal{C}_{\mathbb{T}}$

$$(\varphi \wedge \psi)(\vec{x}) \hookrightarrow \varphi(\vec{x})$$

is covering for the topology $J_{\mathbb{T}'}$.

So the property

$$\varphi \vdash_{\vec{x}} \varphi \wedge \psi$$

is an axiom of the theory $\mathbb{T}_{J_{\mathbb{T}'}}$.

However, it is equivalent to the property

$$\varphi \vdash_{\vec{x}} \psi \, .$$

Verification of the second part of (i): any axiom of $\mathbb{T}_{J_{\mathbb{T}'}}$ is provable in $\mathbb{T}'$.

- By definition, the topology $J_{\mathbb{T}'}$ is generated by the covering morphisms

$$(\varphi \wedge \psi)(\vec{x}) \hookrightarrow \varphi(\vec{x})$$

indexed by the axioms $\varphi \vdash_{\vec{x}} \psi$ of $\mathbb{T}'$.

- We are therefore reduced to proving that the collection of families of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x}))_{i \in I}$$

such that the property

$$\varphi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x})$$

is $\mathbb{T}'$-provable,
is stable under base change and under transitivity.

Stability by base change:

Let us therefore consider a morphism of $\mathcal{C}_{\mathbb{T}}$

$$\theta(\vec{y},\vec{x}) : \psi(\vec{y}) \longrightarrow \varphi(\vec{x}) \,.$$

If the property

$$\varphi \vdash_{\vec{x}} \bigvee_{i\in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i,\vec{x})$$

is provable in $\mathbb{T}'$,
so is the property

$$\psi \vdash_{\vec{y}} \bigvee_{i\in I} (\exists\, \vec{x}_i)(\exists\, \vec{x})\, (\theta_i(\vec{x}_i,\vec{x}) \wedge \theta(\vec{y},\vec{x}))$$

since the property

$$\psi \vdash_{\vec{y}} (\exists\, \vec{x})(\theta(\vec{y},\vec{x}) \wedge \varphi(\vec{x}))$$

is provable in $\mathbb{T}$ and a fortiori in $\mathbb{T}'$.

Stability by transitivity:

Consider a second family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta'_j(\vec{y}_j,\vec{x}) : \psi_j(\vec{y}_j) \longrightarrow \varphi(\vec{x}))_{j\in I'}$$

such that, for any index $i \in I$,

the family which is deduced by the base change morphism

$$\theta_i(\vec{x}_i,\vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x})$$

satisfies the condition that the associated property

$$\varphi_i \vdash_{\vec{x}_i} \bigvee_{j\in I'} (\exists\, \vec{y}_j)(\exists\, \vec{x})\, (\theta'_j(\vec{y}_j,\vec{x}) \wedge \theta_i(\vec{x}_i,\vec{x}))$$

is provable in $\mathbb{T}'$.

For any such $i \in I$, the subobject $\theta_i(\vec{x},\vec{y}) \hookrightarrow \varphi_i(\vec{x}_i) \times \varphi(\vec{x})$ projects on $\varphi_i(\vec{x}_i)$ by an isomorphism, and therefore the property

$$\theta_i \vdash_{\vec{x}_i,\vec{x}} \bigvee_{j\in I'} (\exists\, \vec{y}_j)(\theta'_j(\vec{y}_j,\vec{x}) \wedge \theta_i(\vec{x}_i,\vec{x}))$$

is provable in $\mathbb{T}'$. As

$$\varphi \vdash_{\vec{x}} \bigvee_{i\in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i,\vec{x})$$

is provable in $\mathbb{T}'$, so are $\mathbb{T}'$-provable

$$\varphi \vdash_{\vec{x}} \bigvee_{i\in I} \bigvee_{j\in I'} (\exists\, \vec{x}_i)(\exists\, \vec{y}_j)(\theta'_j(\vec{y}_j,\vec{x}) \wedge \theta_i(\vec{x}_i,\vec{x})) \quad \text{and} \quad \varphi \vdash_{\vec{x}} \bigvee_{j\in I'} (\exists\, \vec{y}_j)\, \theta'_j(\vec{y}_j,\vec{x}).$$

Verification of the first part of (ii): the topology J is contained in the topology $J_{\mathbb{T}_J}$.

Consider a J-covering family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x}))_{i \in I} \, .$$

So the property

$$\varphi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \, \vec{x}_i) \, \theta_i(\vec{x}_i, \vec{x})$$

is an axiom of $\mathbb{T}_J$,
therefore the monomorphism of $\mathcal{C}_{\mathbb{T}}$

$$\varphi(\vec{x}) \wedge \bigvee_{i \in I} (\exists \, \vec{x}_i) \, \theta_i(\vec{x}_i, \vec{x}) \hookrightarrow \varphi(\vec{x})$$

is covering for the topology $J_{\mathbb{T}_J}$.
However, the family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta_{i'}(\vec{x}_{i'}, \vec{x}) : \varphi_{i'}(\vec{x}_{i'}) \longrightarrow \bigvee_{i \in I} (\exists \, \vec{x}_i) \, \theta_i(\vec{x}_i, \vec{x}))_{i' \in I}$$

is covering for the topology $J_{\mathbb{T}_J} \supseteq J_{\mathbb{T}}$,
hence also the family of morphisms

$$(\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x}))_{i \in I} \, .$$

Verification of the second part of (ii): the topology $J_{\mathbb{T}_J}$ is contained in the topology J.

By construction, $J_{\mathbb{T}_J}$ is the topology generated on $J_{\mathbb{T}} \subseteq J$ by the monomorphisms of $\mathcal{C}_{\mathbb{T}}$

$$\varphi(\vec{x}) \wedge \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x}) \hookrightarrow \varphi(\vec{x})$$

associated with the families of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$(\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x}))_{i \in I}$$

which are J-covering
or, which comes to the same thing,
are such that the associated monomorphism

$$\varphi(\vec{x}) \wedge \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x}) \hookrightarrow \varphi(\vec{x})$$

is J-covering.

This ends the proof of the theorem.

The general question of making explicit the correspondence between topologies and quotient theories:

Let us consider in general

- a (small) site $(\mathcal{C}, J)$,
- a geometric theory $\mathbb{T}$,
- an equivalence of toposes

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} .$$

Fact. –
We already know that such an equivalence induces a bijection between

- *the set of topologies J' of $\mathcal{C}$ containing J,*
- *the set of equivalence classes of quotient theories $\mathbb{T}'$ of $\mathbb{T}$.*

Question. –
Is this bijection constructive?
Can we make it explicit?

Program to handle this issue:

→ Given

- a (small) site $(\mathcal{C}, J)$,
- a geometric theory $\mathbb{T}$,

concretely describe the morphisms of toposes

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}} \,.$$

→ Exhibit necessary and sufficient conditions so that such a morphism of toposes

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is an equivalence.

→ Given such a concretely defined morphism

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}}$$

which satisfies the conditions to be an equivalence, describe explicitly and constructively the induced bijection between

- the topologies $J' \supseteq J$ of $\mathcal{C}$,
- the quotient theories $\mathbb{T}'$ of $\mathbb{T}$, up to equivalence.

Description of the morphisms from a topos of sheaves to a classifying topos:

- If $M_{\mathbb{T}}$ is the universal model of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$, the functor

$$\left(\widehat{\mathcal{C}}_J \xrightarrow{(f^*, f_*)} \mathcal{E}_{\mathbb{T}}\right) \longmapsto f^* M_{\mathbb{T}}$$

  is an equivalence

  - from the category of topos morphisms

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}}$$

  - to the category of models of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$

$$\mathbb{T}\text{-mod}(\widehat{\mathcal{C}}_J)\,.$$

- If Σ is the signature of the geometric theory $\mathbb{T}$,

$$\mathbb{T}\text{-mod}(\widehat{\mathcal{C}}_J)$$

  is the full subcategory of that of Σ-structures

$$\Sigma\text{-str}(\widehat{\mathcal{C}}_J)$$

  consisting of the Σ-structures of $\widehat{\mathcal{C}}_J$
  which are models of $\mathbb{T}$ i.e. satisfy its axioms.

Description of models in a topos of sheaves:

- A Σ-structure in $\widehat{\mathcal{C}}_J$ is an application M that associates

  - to any "sort" A of Σ a presheaf
  $$MA : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \quad \text{which is a } J\text{-sheaf},$$
  - to any "function symbol" $f : A_1 \cdots A_n \to B$ a sheaf morphism i.e. presheaf morphism
  $$MA_1 \times \cdots \times MA_n \xrightarrow{Mf} MB\,,$$
  - to any "relation symbol" $R \rightarrowtail A_1 \cdots A_n$ a sub-presheaf
  $$MR \hookrightarrow MA_1 \times \cdots \times MA_n \quad \text{which is a } J\text{-sheaf}.$$

- Any geometric formula φ of Σ of context $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$ interprets in any Σ-structure M of $\widehat{\mathcal{C}}_J$ as a sub-presheaf
$$M\varphi(\vec{x}) \hookrightarrow MA_1 \times \cdots \times MA_n \quad \text{which is a sheaf}.$$

- A Σ-structure M of $\widehat{\mathcal{C}}_J$ is a model of the theory $\mathbb{T}$ if, for any axiom $\varphi \vdash_{\vec{x}} \psi$ of $\mathbb{T}$ of context $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$, we have the inclusion relation between sub-presheaves of $MA_1 \times \cdots \times MA_n$
$$M\varphi(\vec{x}) \subseteq M\psi(\vec{x})\,.$$

Explanation of the interpretation of geometric formulas:

- The interpretation of the geometric formulas of a signature Σ requires

(1) to form products and compose morphisms
to interpret terms,

(2) to form fiber products to interpret atomic formulas,

(3) to form fiber products of subobjects
to interpret the symbols $\wedge$,

(4) to form images of morphisms $E' \to E$
i.e. colimits of diagrams

$$E' \times_E E' \rightrightarrows E'$$

to interpret the symbols $\exists$,

(5) to form unions of subobjects $E_i \hookrightarrow E$
i.e. colimits of diagrams

$$\coprod_{i,j} E_i \times_E E_j \rightrightarrows \coprod_i E_i$$

to interpret the symbols $\vee$ or $\bigvee$.

Interpretation in presheaves:

- Thus,

  - the interpretation of (1), (2) and (3), i.e. atomic formulas and Horn formulas, is done in terms of composition of morphisms and finite limits,
  - the interpretation of (4) and (5), i.e. regular, coherent or geometric formulas, is done in terms of finite limits and arbitrary colimits.

- In the topos $\widehat{\mathcal{C}}$ of presheaves $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, these interpretations are made component by component since the evaluation functors at the objects X of $\mathcal{C}$

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \longrightarrow & \mathrm{Set}\,, \\ P & \longmapsto & P(X) \end{array}$$

  respect both limits and colimits.

Interpretation in sheaves:

- The embedding functor

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

respects limits, while the sheafification functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J$$

respects finite limits and colimits,
and the composite $j^* \circ j_*$ identifies with $\mathrm{id}_{\widehat{\mathcal{C}}_J}$.

- Therefore,
  - the interpretation of (1), (2) and (3),
    i.e. of atomic formulas and Horn formulas,
    is the same in $\widehat{\mathcal{C}}_J$ as in $\widehat{\mathcal{C}}$
    therefore is realised component by component,
  - the interpretation in $\widehat{\mathcal{C}}_J$ of (4) and (5),
    i.e. of regular, coherent or geometric formulas,
    is done in two steps:
    first in $\widehat{\mathcal{C}}$, i.e. component by component,
    then by applying the sheafification functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J .$$

Under what conditions is a model universal?

We consider a model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$
which corresponds to a topos morphism

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}} \, .$$

Question. – *Under what conditions is the morphism*

$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}}$$

an equivalence ?

Situation. – $\mathcal{E}_{\mathbb{T}}$ can be constructed as the topos of sheaves on

$$(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

where

- $\mathcal{C}_{\mathbb{T}}$ is the syntactic category in a fragment of logic which can be

$$\begin{cases} \text{geometric} & \text{in the general case of a geometric theory } \mathbb{T}, \\ \text{coherent} & \text{if } \mathbb{T} \text{ is a coherent theory}, \\ \text{regular} & \text{if } \mathbb{T} \text{ is a regular theory}, \\ \text{Cartesian} & \text{if } \mathbb{T} \text{ is a Cartesian theory}, \end{cases}$$

- $J_{\mathbb{T}}$ is the syntactic topology of geometric [resp. coherent, resp. regular, resp. discreet] type.

First necessary condition:

For the model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$ to be universal, the following is necessary:

Condition (A). – *Going through the family of objects of $\mathcal{C}_{\mathbb{T}}$ i.e. of formulas*

$$\varphi(\vec{x}) \quad \text{in a context } \vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

which are geometric
[resp. coherent, resp. regular, resp. $\mathbb{T}$-cartesian],
the family of their interpretations in M

$$M\varphi(\vec{x}) \hookrightarrow MA_1 \times \cdots \times MA_n$$

must be separating as a family of objects of $\widehat{\mathcal{C}}_J$.

Remarks. –

(i) This means that for any pair of morphisms of $\mathcal{C}$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

whose images by $\ell : \mathcal{C} \to \widehat{\mathcal{C}}_J$ are distinct,
there must exist a formula $\varphi(\vec{x})$ and a morphism $\quad M\varphi(\vec{x}) \xrightarrow{m} \ell(X)$
such that $\ell(f) \circ m \neq \ell(g) \circ m$.

(ii) In (i), we can replace the $M\varphi(\vec{x})$
by the interpretations of the $\varphi(\vec{x})$ in $\widehat{\mathcal{C}}$.

Second necessary condition:

For the model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$ to be universal, the following is necessary:

Condition (B). – *For a family of morphisms of $\mathcal{C}_{\mathbb{T}}$*

$$\theta_i(\vec{x}_i, \vec{x}) : \varphi(\vec{x}_i) \longrightarrow \varphi(\vec{x})$$

to be $J_{\mathbb{T}}$-covering, (it is necessary and) it suffices that the image morphism in $\widehat{\mathcal{C}}_J$

$$\coprod_{i \in I} M\varphi_i(\vec{x}_i) \longrightarrow M\varphi(\vec{x}) \quad \text{be an epimorphism.}$$

Remarks. –

(i) The image morphism is an epimorphism when $M\varphi(\vec{x})$ is the transform by j^* of the presheaf

$$\mathrm{Ob}(\mathcal{C}) \ni X \longmapsto \bigcup_{i \in I} \mathrm{Im}(M\varphi_i(\vec{x}_i)(X) \to M\varphi(\vec{x})(X))\,.$$

(ii) Such a family of morphisms of $\mathcal{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering when

- in the geometric case

$$\varphi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x}) \text{ is } \mathbb{T}\text{-provable,}$$

- in the coherent case, there exists $i_1, \cdots, i_n \in I$ such that

$$\varphi \vdash_{\vec{x}} (\exists\, \vec{x}_{i_1})\, \theta_{i_1}(\vec{x}_{i_1}, \vec{x}) \vee \cdots \vee (\exists\, \vec{x}_{i_n})\, \theta_{i_n}(\vec{x}_{i_n}, \vec{x}) \text{ is } \mathbb{T}\text{-provable,}$$

- in the regular [resp. Cartesian] case, there exists $i_0 \in I$ such that

$$\varphi \vdash_{\vec{x}} (\exists\, \vec{x}_{i_0})\, \theta_{i_0}(\vec{x}_{i_0}, \vec{x}) \text{ is } \mathbb{T}\text{-provable,}$$

  [resp. the identity of $\varphi(\vec{x})$ factors through $\varphi_{i_0}(\vec{x}_{i_0}) \xrightarrow{\theta_{i_0}(\vec{x}_{i_0}, \vec{x})} \varphi(\vec{x})$].

Third necessary condition:

For the model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$ to be universal, the following is necessary:

Condition (C). – *For any* <u>*pair of objects*</u> *of* $\mathcal{C}_{\mathbb{T}}$

$$\varphi(\vec{x}) \qquad \textit{and} \qquad \psi(\vec{y})\,,$$

and for any <u>*morphism*</u> *of* $\widehat{\mathcal{C}}_J$ *between their* <u>*interpretations*</u> *in* M

$$M\varphi(\vec{x}) \xrightarrow{\ u\ } M\psi(\vec{y})\,,$$

there must exist a $J_{\mathbb{T}}$*-*<u>*covering*</u> *family of morphisms of* $\mathcal{C}_{\mathbb{T}}$

$$\theta_i(\vec{x}_i,\vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x})\,, \quad i \in I\,,$$

and a family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$\theta'_i(\vec{x}_i,\vec{y}) : \varphi_i(\vec{x}_i) \longrightarrow \psi(\vec{y})\,, \quad i \in I\,,$$

making <u>*commutative*</u> *the triangles of* $\widehat{\mathcal{C}}_J$*:*

$$\begin{array}{ccc} M\varphi_i(\vec{x}_i) & & \\ {\scriptstyle M\theta_i}\big\downarrow & \searrow{\scriptstyle M\theta'_i} & \\ M\varphi(\vec{x}) & \xrightarrow{\ u\ } & M\psi(\vec{y}) \end{array}$$

Note. – To check the commutativity of these triangles, it is enough <u>to evaluate the sheaves</u> $M\psi(\vec{y})$, $M\varphi(\vec{x})$ and $M\varphi_i(\vec{x}_i)$ at the objects X de $\mathcal{C}$.

Necessary and sufficient conditions:

For the model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$ to be universal,
the previous necessary conditions are sufficient:

Proposition. –
In order for a model M of a geometric theory $\mathbb{T}$ *in the topos* $\widehat{\mathcal{C}}_J$ *of sheaves on a site* $(\mathcal{C}, J)$ *to define an equivalence of toposes*

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \,,$$

it is necessary and it suffices that M verifies conditions (A), (B) and (C) above.

Proof. –
We apply Corollary 5.11 of the prepublication:

O. Caramello, *"Denseness conditions, morphisms and equivalences of toposes"* (2020).

Topologies associated with a quotient theory:

We consider an equivalence of toposes

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

defined by a model M of $\mathbb{T}$ in a topos of sheaves $\widehat{\mathcal{C}}_J$.

Proposition. –

Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$,
defined by adjoining to the axioms of $\mathbb{T}$
extra axioms $\varphi_i \vdash \psi_i$, $i \in I$.
Let J' the unique topology of $\mathcal{C}$ containing J
which induces an equivalence of toposes

$$\widehat{\mathcal{C}}_{J'} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}'} .$$

Then J' is the topology generated on J by the sieves

$$y(X) \times_{M\varphi_i} M(\varphi_i \wedge \psi_i) \hookrightarrow y(X) \qquad (\textit{where } y : \mathcal{C} \to \widehat{\mathcal{C}} \textit{ is Yoneda})$$

associated with

- *axioms $\varphi_i \vdash \psi_i$, $i \in I$,*
- *objects X of $\mathcal{C}$,*
- *elements of $M\varphi_i(X)$ seen as morphisms of $\widehat{\mathcal{C}}$*

$$y(X) \longrightarrow M\varphi_i .$$

Remark. – The family of sieves

$$y(X) \times_{M\varphi_i} M(\varphi_i \wedge \psi_i) \hookrightarrow y(X)$$

is stable under pull-back by the morphisms $X' \to X$ of $\mathcal{C}$.
It is therefore the same for its union with J.
To transform this union into the topology J',
it suffices to form all the multicomposites of covering families.

Proof of the proposition. –
For a topology K of $\mathcal{C}$ containing J, the sheafification functor

$$\widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J \longrightarrow \widehat{\mathcal{C}}_K$$

transforms into isomorphisms of $\widehat{\mathcal{C}}_K$ all embeddings of $\widehat{\mathcal{C}}_J$

$$M(\varphi_i \wedge \psi_i) \hookrightarrow M\varphi_i$$

if and only if all the sieves of the form

$$y(X) \times_{M\varphi_i} M(\varphi_i \wedge \psi_i) \hookrightarrow y(X)$$

are K-covering.
So J' is necessarily the smallest of topologies K
that satisfy these conditions.

Quotient theories that correspond to a topology:

We still consider an equivalence of toposes

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} ,$$

for a geometric theory $\mathbb{T}$ of signature Σ,
defined by a model M of $\mathbb{T}$ in a topos of sheaves $\widehat{\mathcal{C}}_J$.

Proposition. – *Let J' be a topology of $\mathcal{C}$ that contains J.*
Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$ such that the equivalence

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

induces an equivalence

$$\widehat{\mathcal{C}}_{J'} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}'} .$$

Then a property linking geometric formulas of Σ

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}'$ if and only if, for any object X of $\mathcal{C}$
and any element of $M\varphi(X)$ seen as a morphism of $\widehat{\mathcal{C}}$

$$y(X) \longrightarrow M\varphi ,$$

the sieve

$$y(X) \times_{M\varphi} M(\varphi \wedge \psi) \hookrightarrow y(X)$$

is an element of J'.

Proof. – The topos $\mathcal{E}_{\mathbb{T}'}$ is a subtopos of $\mathcal{E}_{\mathbb{T}}$
i.e. they are related by a topos morphism

$$\left(\mathcal{E}_{\mathbb{T}} \xrightarrow{e^*} \mathcal{E}_{\mathbb{T}'}, \mathcal{E}_{\mathbb{T}'} \overset{e_*}{\hookrightarrow} \mathcal{E}_{\mathbb{T}}\right)$$

which is an embedding in the sense that e_* is fully faithful.
The functor e^* transforms
the universal model of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$
into the universal model of $\mathbb{T}'$ in $\mathcal{E}_{\mathbb{T}'}$,
and it respects interpretations of geometric formulas.
A property

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}'$ if and only if the embedding of $\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}}$

$$M(\varphi \wedge \psi) \hookrightarrow M\varphi$$

is transformed by e^* into an isomorphism of $\widehat{\mathcal{C}}_{J'} \cong \mathcal{E}_{\mathbb{T}'}$.
This amounts to requiring that for any object X of $\mathcal{C}$
and any morphism $y(X) \longrightarrow M\varphi$,
the sieve

$$y(X) \times_{M\varphi} M(\varphi \wedge \psi) \hookrightarrow y(X)$$

be an element of J'.

Application to provability:

We consider as before an equivalence of toposes

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

defined by a model M of $\mathbb{T}$ in a topos of sheaves $\widehat{\mathcal{C}}_J$.

Corollary. – *Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$, defined by adjoining to the axioms of $\mathbb{T}$ extra axioms*

$$\varphi_i \vdash \psi_i, \qquad i \in I\,.$$

Then a geometric property of the form

$$\varphi \vdash \psi$$

is provable in $\mathbb{T}'$ if and only if the sieves of the form

$$M(\varphi \wedge \psi) \times_{M\varphi} y(X) \hookrightarrow y(X)$$

can be obtained by multicomposition in $\mathcal{C}$ of sieves of J and sieves of the form

$$M(\varphi_i \wedge \psi_i) \times_{M\varphi_i} y(Y) \hookrightarrow y(Y)\,.$$

Proof. – It suffices to combine the two preceding propositions.